MODELLING OF J-CURVE AND S-CURVE USING DIFFERENTIAL EQUATIONS: STUDIES IN ECONOMICS, ENTREPRENEURSHIP AND FINANCE

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ABSTRACT

MODELLING OF J-CURVE AND S-CURVE USING DIFFERENTIAL EQUATIONS:

STUDIES IN ECONOMICS, ENTREPRENEURSHIP AND FINANCE

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J-Curve phenomenon, which shows how a system responds to an external influence, is observed in many areas like economics, financial investments, healthcare, etc.

We propose a mathematical formulation and framework for the J-Curve in terms of the Riccati differential equation and its associated Laguerre equation. The solutions describing the J-Curve are set up as polynomials similar to the Laguerre polynomials.

We give explicit functional forms for the system characteristics if it has to manifest J-Curve behavior and provide physical interpretations of the various terms in the Riccati equation to help understand the characteristics of any system manifesting a J-Curve behavior. We also set up criteria for any curve to be mathematically validated as a J-Curve.

The Riccati differential equation is used to describe the S-Curve, which describes the cumulative sales growth or population dynamics. Thus, the Riccati equation is shown to unify the mathematical basis of S-Curve and J-Curve.

We analyze five case studies for J-Curve behavior under the defined mathematical framework – a) four parameters of Indian economy are studied from 1960-2020 to validate J-Curve phenomenon post economic liberalization in 1991, b) Internal Rate of Returns for

venture investments are proven to exhibit J-Curve, c) long term investments in stock markets are shown to follow J-Curve, d) The GDPs of some countries/regions post the 2007-08 financial crisis are analyzed for J-Curve, 5) GDP of Croatia is shown to exhibit J-Curve post-independence, as well as post global financial crisis.

An interesting property of the Riccati differential equation is also shown to explain the pharmacokinetic absorption of medicines in the body.

This is the first time 1) a mathematical formalism is set up to define the J-Curve phenomenon, 2) an explicit differential equation is defined for the J-Curve, 3) the functional forms of the system's inertia, environmental damping, as well as the external influence acting on it are given, 4) explicit equation (polynomials with alternating coefficients) which shows the J-Curve behavior is described, 5) the mathematical conditions any curve has to satisfy if it has to qualify as a J-Curve are highlighted, 6) the S-Curve and J-Curve are both shown to be special cases of the generic nonlinear 1st order Riccati differential equation, 7) functional form of external influence on a system to manifest J-Curve behavior is explicitly discussed in the context of pharmacokinetics, and the functional form of medicine absorption in the body is presented.

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CHAPTER I:

INTRODUCTION

1.1 Diffusion of Innovation, S-Curve, J-Curve – Background

Inventions, innovations and discoveries have been associated with humans for centuries. "Necessity is the Mother of Invention" is a very often quoted saying ---- meaning when there is a need, new ideas/methods are invented or discovered, to overcome an existing problem.

Some of the well-known inventions or discoveries from ancient times are the fire, the wheel, and weapons for hunting. After the fire was discovered, weapons graduated from bones to metals. People had started working with the pulleys, fulcrum, inclined planes, chariots, slingshots, ships, and other engineering applications. Medicines were discovered, chemicals were used in fireworks and weapons, and metallurgy was well studied and applied.

The past few hundred years saw rapid growth of knowledge and its various applications ---- the industrial revolution, the steam and internal-combustion engines, the printing press, transportation via railways, basic sciences like physics (gravity, electromagnetism), mathematics (calculus, differential equations), chemistry (elements and compounds), engineering (civil, mechanical, metallurgical).

In the modern era (approximately past 200 years) there has been an explosion of innovations, discoveries and inventions on multiple fronts ---- electromagnetism (power, lighting), medicine (important drugs, medical diagnostics, gene technology), flight (planes and space missions), atomic energy (peaceful and weapons), electronics (digital revolution), computing (mainframe and desktops), telecom (telegraph, wireline/wireless telephony, fibre optics), internet (ubiquitous), communication (voice, data, video), nanotechnology (merging various disciplines of sciences and engineering for unique applications), healthcare (diagnostics, treatments), alternative sources of energy (solar, wind, bio), white goods (fridge, microwave, TV), economics (financial modelling, stock markets), education (digital outreach, online courses and programs), etc.

Earlier the innovations were driven by “necessity” ---- there was a problem looking for a solution (social entrepreneurship is an ideal example of this class, with societal problems looking for an answer). With advances in knowledge and technology, there was another vector to the innovations which were “opportunity” driven ---- there was a solution looking for a problem (the invention of the transistor in an example in this category, a technology which addressed many problems in electronics, computing and telecommunication). A very good study of these two “necessity” and “opportunity” approaches across many countries have been studied (der Zwan *et al.*, 2016).

Innovations gradually became more organized in the 19^{th} and 20^{th} century, setting the platform for Intellectual Property Rights (IPR) concepts like patents, trade secrets, trademarks, etc. Protection of IP got related to its impact on business and economic growth.

Innovations have arisen from the entrepreneurial spirit of the inventors – whether they be individuals, startups, or organizations. Benefits of some form have been the key drivers of innovations – social, technological, cost, new applications or services, economical, financial, etc. Profits and financial benefits coupled with wealth creation have driven many of the recent innovations (social entrepreneurship may be an exception).

Formal studies in entrepreneurship started in the 20^{th} century – “**Diffusion of Innovation**” (DoI) being one of the most important concepts studied in the theory of economic development (Rogers, 2003). An important aspect of innovations becoming useful is how they are accepted by the targeted end customers, hence allowing the innovation to “diffuse” amongst the people. There are different stages in which innovation gets “diffused” into the population, and each stage is characterized by different types of population which adopt the innovation. Diffusion-of-Innovation results in the market acceptability of the innovation, and explains how any new idea/product/service gets ingrained into mainstream society – which in turn results in the cumulative acceptance of the innovation, and leads to growth of the business, income and financial returns, and customer base.

Meade and Islam (2006) have given a 25-year review of the Diffusion-of-Innovations in various perspectives, and also the relation to the **Cumulative-Growth-Curve** (depicted by the **S-Curve**). Dearing (2009) studied the Diffusion-of-Innovation in the context of social work.

Along with the Diffusion-of-Innovation and the S-Curve, another important and associated concept when studying startups and their evolution, is the **Cash Flow Curve** (depicted by the **J-Curve)**. The J-Curve has much wider applications and implications in various domains like healthcare, trade policies, financial investments and economies of countries, etc.

In this thesis, the J-Curve will be studied in great detail and a mathematical formalism and model will be provided. The differential equation for J-Curve will be given in terms of the Riccati differential equation. Explicit functional form of the J-Curve will be provided in terms of polynomials similar to the Laguerre polynomials. The S-Curve will also be modelled as a Riccati differential equation. Hence, both the S-Curve and the J-Curve will be put on a common mathematical platform.

1.2 Some Features of S-Curve and J-Curve

The S-Curve has the typical shape of a **Sigmoid-Function** (Yin *et al.*, 2002)**.** This curve has been studied for a long time across many disciplines (population, sales, etc.). It is also called the **Logistics-Curve**. The study of S-Curve and its various aspects and applications have been well studied (Runge, 2014). The initial applications of the S-Curve have been given in the context of population growth. Bacaër (2011a) gives a brief description of the various methods and techniques used in the study of population growth over many centuries.

Along with the Cumulative-Growth-Curve, another very important aspect of a startup financial analysis is the Cash-Flow-Curve (CFC). Sometimes the Cash-Flow-Curve is also referred to as the **Profit-Curve** or **Growth-Curve**, and is related to the more generic **J-Curve**. It shows how the Cash Flow (CF) varies in the start/growth/evolution phases of a startup from inception to maturity – and gives a good description of the net cash flow. The Cash-Flow-Curve is negative (cash outflow more than inflow) in the beginning, then becomes positive (as sales pick up) to eventually stabilize (as the product becomes mature). Dushca and Davidavičienė (2016) show how the S-Curve (or Cumulative-Growth-Curve), and J-Curve (or Cash-Flow-Curve) are related. Strauss (2014) shows the Growth-Curve in terms of funding stages, and shows how it follows the J-Curve. Love (2016) gives a good description J-Curves in the context of the evolution of startups through various stages.

J-Curve has implications and importance beyond the startup ecosystem. Backus *et al.* (1994) look at the J-Curve in the context of Trade Balance and Capital for various countries, while Carter and Pick (1989) study the J-Curve in the context of the relation of Trade Balance with Currency Depreciation. Bremmer (2006) discusses the rise and fall of nations using the J-Curve in the context of Stability and Openness. Kebbi et al (cited in

(Minna Tunkkari Eskelinen and Iiris Aaltio, 2016) talk about the Double-J-Curve. Dickens and Solomon (1938) explain the importance of Conformity Behavior in terms of J-Curve.

A characteristic feature of the J-Curve is that it is a manifestation of the "cause-and-effect" phenomenon – the effect of any cause is seen after a lag. When a system is acted upon by any change, it takes a certain time lag before the effect of that cause manifests itself – e.g. i) trade imbalance has a time lag before the effect of the currency depreciation kicks in, ii) private equity funds give a negative return in the initial stages before the giving positive Internal Rate of Return (IRR), iii) the GDP of countries take time to change after the policies changes are implemented, iv) patients respond to medication after a time lag. Hence the cause-and-effect phenomenon of the J-Curve is very important, and it is desirable to have a formal and mathematical foundation to explain it.

Thus the importance of the study of S-Curve, as well as the J-Curve, cannot be over emphasised.

The Cumulative-Growth-Curve aspect of the J-Curve has been empirically studied in 1830s by Quetelet, also by Verlhust (cited in (Bacaër, 2011a)). Pearl and Reed (1920) gave a mathematical model in the 1920s. Diffusion-of-Innovation came in the 1950s, and this led to the idea of the Bell-Curve, Cumulative-Growth-Curve (which is also the S-Curve, and which is the cumulative effect of the Bell-Curve (Rogers, 2003), Cash-Flow-Curve, etc. <u>The J-Curve does not seem to have been given a formal mathematical basis, nor has there been any mathematical model describing it, or giving it a functional form. This thesis will address this problem.</u>

1.3 Gaps in the Present Knowledge

a. Very little work has been done in the mathematical analysis of the S-Curve and J-Curve.

b. No mathematical formulation, or equation, exists for the J-Curve.

c. No model exists to show how the role of the ecosystem parameters can quantitatively and mathematically play a role in the evolution of the J-Curve.

d. There is no formalism to bring out a common mathematical platform to unify and S-Curve and J-Curve.

e. There is no quantitative framework which can mathematically state if a curve is actually a J-Curve, or "just looks like a J-Curve".

f. The relevance and importance of mathematical modelling in entrepreneurship using differential equations has not been studied before.

1.4 Research Questions

Some of the questions which drive this research are:

A. What formal methods have been used to study S-Curve and J-Curve?

B. Is there a common mathematical basis and formalism to describe the S-Curve and the J-Curve?

C. What are the important parameters that physically play a role in determing the dynamics of J-Curve?

D. How the important ecosystem parameters like startup organization, funding, policies, market, customers, competition, etc., can be comprehended while studying the dynamics of J-Curve in startups, economics, finance, etc.?

E. Will differential equations play an important role in studying important concepts in entrepreneurship? If yes, are there particular class of differential equations which are relevant?
F. Will modelling via differential equations help find analytical solutions/equations for the J-Curve?
G. Will modelling via differential equations give a different insight into the evolution and dynamics of startups?
H. Is possible to establish quantifiable mathematical criteria for any phenomenon to be classified as a J-Curve?

1.5 Purpose of Research

There are multiple purposes for carrying out this research:

i. Analyze and study the S-Curve and the J-Curve to illustrate how formal mathematical tools like differential equations can be used to study important aspects of entrepreneurship from form novel perspectives, and come to new and interesting conclusions.
ii. Illustrate how physical understanding of the various ecosystem aspects/parameters, along with a good domain knowledge, are necessary to set up good mathematical models.
iii. Give the differential for the J-Curve.
iv. Show that the exact polynomial equation that describes the J-Curve can be given in terms of a famous orthogonal polynomial – the Laguerre polynomial.
v. Give a common mathematical platform for describing the S-Curve, as well as the J-Curve in terms of the Riccati differential equation (one of the most

important first-order nonlinear differential equation in physics and engineering).

vi. Establish quantifiable mathematical criteria for any phenomenon to be classified as a J-Curve.

vii. Show that the mathematical formalism for J-Curve is applicable across various disciplines like economics, finance, stock markets, venture investments, medicine and healthcare, etc.

viii. Show that formal methods can bring different perspectives into the study of various domains and subjects, and help to get an understanding of the subject.

ix. Show that innovation and entrepreneurship is a domain, which is fertile to adopting formal mathematical methods, like differential equations. This should open the doors for many more, and other, formal mathematical tools to be applied in the domain of entrepreneurship.

1.6 What will be studied in this thesis?

In this thesis, we will study and analyse the mathematical foundations of the **S-Curve** (**Cumulative-Growth-Curve** is an illustrative example), and **J-Curve** (**Cash-Flow-Curve** is an illustrative example). We will show that previous studies have been done of the S-Curve, and equations exist to describe them - these will be briefly summarized and explained to set the context.

We reformulate the equations of the S-Curve in a more generic setting of the Riccati differential equation. We also explore and study the platform of the Riccati differential equation to see if the J-Curve can also be represented by a Riccati differential equation.

We provide mathematical models for both the S-Curve as well as the J-Curve in terms of the Riccati differential equation, explore various forms of the Riccati differential equations and propose specific Riccati differential equations which are suitable to model the S-Curve and the J-Curve.

This study will find and show that the Riccati differential equation (Davis, 1975) is the most suitable mathematical equation to describe the S-Curve, as well as the J-Curve. We show the solution of the Riccati differential equation for the J-Curve to be in the form of one of the important orthogonal polynomials - the Laguerre Polynomials (Abramowitz and Stegun, 1972). We bring out and explain the physical significance of the different mathematical terms in these equations, along with their relations to various aspects of the startup and entrepreneurial ecosystem.

Functional form of external influence on a system to manifest J-Curve behavior is explicitly discussed in the context of pharmacokinetics, and the functional form of medicine absorption in the body is presented.

1.7 Significance of the Study

There are many significant advantages to this research:

1. Study of important concepts in entrepreneurship, economics, finance, etc. like S-Curve (e.g. Cumulative-Growth-Curve), and J-Curve (e.g. Cash-Flow-Curve) using formal mathematical methods.
2. We will show for the first time that:
    a. An important differential equation – the Riccati equation – can be used used as a common platform to give a mathematical basis for the S-Curve, as well as the J-Curve.
    b. Differential equation for the J-Curve is given.
    c. The J-Curve can be described in terms of the famous Laguerre Polynomial (an important orthogonal polynomial) which is the solution of the Riccati

differential equation, and the J-Curve can be described by a polynomial equation with coefficients alternating in sign.

d. It is possible to establish mathematical criteria for any phenomenon to be quantitatively classified as a J-Curve.

e. Ecosystem parameters (e.g. startups, competition, customers and markets, environment, funding and financing, etc.) can be identified and properly interpreted in the mathematical model.

f. It is possible to identify system and ecosystem parameters that determine the evolution of Cash-Flow-Curve in most startups.

g. That various disparate disciplines like finance, economics, medicine and health, startups, etc. which manifest J-Curves, can all now be quantitatively studied with precise mathematical equations under the same mathematical platform.

3. Make both the communities – mathematicians, as well as researchers studying entrepreneurship, economics, healthcare, politics (and many other domains) – aware that formal mathematical methods are very useful tools, and are also naturally applicable, to studying various dynamical aspects of entrepreneurial, economic and financial systems.

CHAPTER II:

REVIEW OF LITERATURE

2.1 History and Importance of the S-Curve

As described in the previous chapter, "Diffusion of Innovation" (DoI) is one of the most important concepts studied in the theory of economic development (Rogers, 2003). This studies how the various innovations are accepted by the targeted end-customers. There are different stages in which innovation gets "diffused" into the population, and each stage is characterized by different types of population which adopt the innovation – a) innovators, b) early adopters, c) early majority, d) late majority, e) laggards.

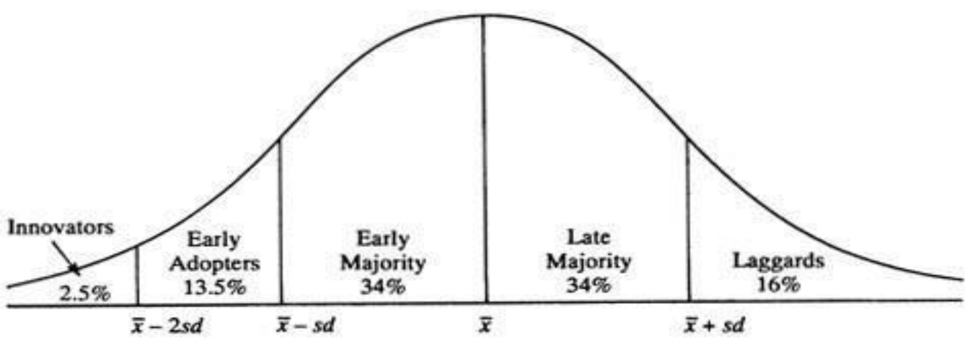


Figure 2.1 *Adopters of Innovation (Rogers, 2003, Fig. 7.3)*

This the typical Bell-Curve for Diffusion-of-Innovation, as adopted by various categories of population. The "early majority" and "late majority" lie on either side of the mean up to 1-sigma (standard deviation); the "early adopters" and "laggards" lie between 1-sigma and 2-sigma; the innovators lie beyond 2-sigma in the early stages.

The Cumulative-Growth-Curve of adoption is the typical S-Curve:

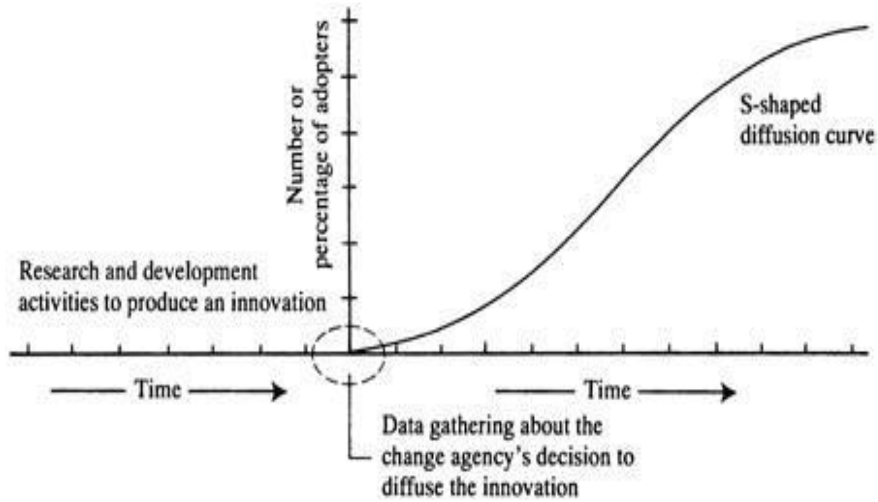


Figure 2.2 *Adopters of Innovation (Rogers, 2003, Fig. 3.3)*

The relation between the Bell-Curve and the S-Curve can be clearly seen from the illustrative example of farmers adopting hybrid corn, in the below figure:

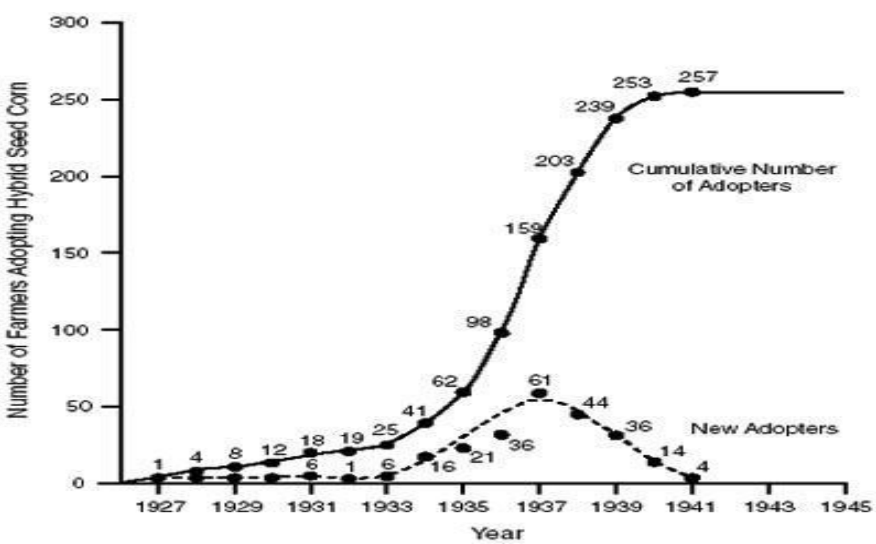


Figure 2.3 *Adopters of Innovation (Rogers, 2003, Fig.7.1)*

Diffusion-of-Innovation results in the market acceptability of the innovation, and explains how any new idea/product/service gets ingrained into mainstream society – which in turn results in the cumulative acceptance of the innovation, and leads to growth of the business, income, and customer base.

(Bacaër, 2011a) has given brief descriptions of the studies of population growth and dynamics by various people over the centuries from various perspectives – Fibonacci (growth of rabbits in 1202), Halley (mortality and relation to annuities in 1693), Euler (geometric growth of population in 1748) and Bernoulli (relation of smallpox, inoculation and mortality in 1760), Malthus (relation between population growth and limited resources in 1798), Quetelet (damping/resistance of limited resources proportional to the square of the population in 1835). In 1838, Verhulst (cited in Bacaër, 2011b) took the idea of Quetelet and wrote the evolution of population with limited resources as a differential equation for the first time. Pritchett (cited in Pearl and Reed, 1920)) gave the cubic algebraic equation for the population curve. This was modified by Pearl and Reed (1920) where the cubic term of Pritchett was replaced by a logarithmic term. The growth bacteria in a given medium follows the S-Curve (Orbit Biotech, 2018), (Tamer and Toprak, 2017). In 1903, Tarde (cited in Burton, 2017) was the first to plot the Diffusion-of-Innovation in terms of the S-Curve.

Asthana (1995) shows a clear relationship between the Bell-Curve and S-Curve. Kucharavy and De Guio (2011) clearly show the relation between the Bell-Curve and S-Curve across the various stages of growth. Schramm (2017) shows the various stages of technology development, which play a role in how the technology-maturity evolves over time, and how to jump from one S-curve to the next. Runge (2014) shows how different technological products like telephone, TV, automobiles, microwave, PCs, cell phones, internet have all depicted the S-Curve in their adoption/growth (only the time scales of adoption vary). He also talks about the “chasm” and its role//place in the Bell-Curve of technology adoption:

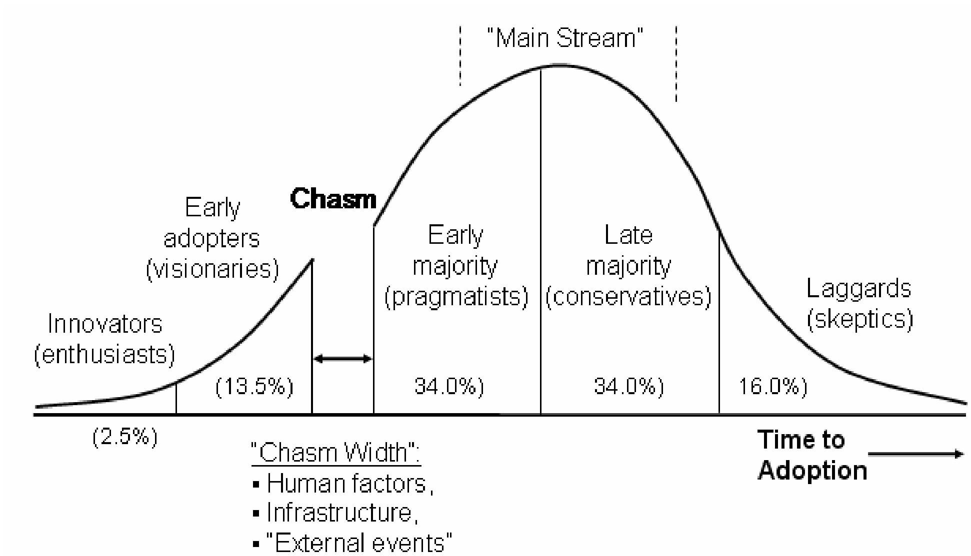


Figure 2.4 *Chasm and Adoption of Innovation (Runge, 2014, Fig.1.29)*

The Cumulative-Growth-Curve shows the growth (of business, population, sales, finance, etc.) and is a cumulative growth of what the Bell-Curve depicts. This curve has been studied for a long time across many disciplines (population, sales, etc.). The Cumulative-Growth-Curve has the typical shape of a Sigmoid-Function (it is also called the Logistics-Curve or S-Curve). The initial applications of the S-Curve have been in the context of population growth - Bacaër (2011a) gives a brief description of the various methods and techniques used in the study of population growth over many centuries. Yin *et al.)* (2002) studied the various mathematical forms using curve fitting. The study of S-Curve and its various aspects and applications have been well studied (Runge, 2014). Dearing (2009) studied Diffusion-of-Innovation in the context of social work.

2.2 Mathematical Aspects of the S-Curve until date

The historical evolution of the mathematical aspects of the Cumulative Growth Curve, which is related to the S-Curve, are now presented.

In 1838, Verhulst (cited in (Bacaër, 2011b)) took the idea of Quetelet and wrote the evolution of population with limited resources as a differential equation for the first time). In its simple form, the equation was

$$\frac{dP(t)}{dt} = A * P(t) * \left(1 - \frac{P(t)}{B}\right) = A * P(t) - A * \frac{P(t)^2}{B} \qquad 2.1$$

where dP/dt denotes the derivative of P with respect to t, A and B are constants, and P^2 is the "negative quadratic" term representing restricted resources. This was the very first time that the study of population dynamics was modelled via a differential equation, in 1838. The solution to this equation is written as:

$$P(t) = \frac{[P(0) * e^{At}]}{\left[1 + P(0) * \frac{e^{At} - 1}{B}\right]} \qquad 2.2$$

Verhulst called the above function the Logistic-Curve, which is the classic S-Curve. He is the first person to introduce a differential equation for population dynamics, and the first to give the solution in terms of the S-Curve – he used it to study the population of Belgium. Thus the S-Curve came to represent the growth of any entity in a restricted environment of limited resources, with the "damping' due to environment given by the "negative quadratic" term. Li *et al.* (2016) show that the mathematical equation for bactrial growth in a medium follows the S-Curve.

Pritchett (1891, cited in (Pearl and Reed, 1920)) gave the following cubic algebraic equation for the population curve

$$P(t) = A + B * t + C * t^2 + D * t^3 \qquad 2.3$$

Pearl and Reed (1920) modified this to

$$P(t) = A + B * t + C * t^2 + D * \log(t) \qquad 2.4$$

where the cubic term of Pritchett was replaced by a logarithmic term.

2.3 History and Importance of the J-Curve

The phenomenon applies in a variety of fields such as economics, medicine, and political science.

Along with the S-Curve, another very important aspect of a startup financial analysis is the Cash-Flow-Curve (CFC). Sometimes the Cash-Flow-Curve is also referred to as the Profit-Curve or Growth-Curve**.** It is also referred to as the J-Curve. Dushca and Davidavičienė (2016) show how the Bell-Curve (or Sales-Curve), S-Curve (or Cumulative-Growth-Curve), J-Curve (or Cash-Flow-Curve) are related. Strauss (2014) illustrates the Growth-Curve in terms of funding stages, and shows that it follows the J-Curve. Love (2016) gives a good description J-Curves in the context of the evolution of startups through various stages.

J-Curves have been recognized to have implications and importance in a variety of fields such as private equity funds, economics, medicine, and political science. Backus *et al. (*1994) look at the J-Curve in the context of Trade Balance and Capital, Carter and Pick

(1989) study the J-Curve in the context of the relation of Trade Balance with Currency Depreciation for various countries, Ahtiala (1983) talks of the J-Curve in the relation between Trade Balance and exchange rates. Bremmer (2006) discusses the rise and fall of nations using the J-Curve in the context of Stability and Openness. Hashim *et al.* (2011) and Virmani and Hashim (2011) studied the impact of India's economic liberalization on productivity, and its relation to the J-Curve, Bahmani-Oskooee (1985) has studied the relation between Devaluation and Trade Balance and demonstrates that the J-Curve is a manifestation of a "lag between cause and effect" – there is usually an initial deterioration followed by an improvement. Hussain and Haque (2014) studied 49 developing countries and showed the J-Curve effect in the relation between Trade Balance and Net Export as a function of Exchange Rates. Kebbi et al (cited in Minna Tunkkari Eskelinen and Iiris Aaltio, (2016)) talk about the Double-J-Curve. Dickens and Solomon (1938) showed the importance of Conformity Behavior in terms of J-Curve. The change in Heart-Rate-Variability as a function of training over time (Plews *et al.*, 2013) and the variability of blood pressure over time (Banach and Aronow, 2012) in response to medication, also seem to show the characteristics of a J-Curve. Dudenbostel and Oparil (2012) observed the J-Curve in the relation between blood pressure and cardiovascular disease.

In the context of startups, an important manifestation of the J-Curve is the Cash Flow (CF). This varies in the start/growth/evolution phases of a startup from inception to maturity – and gives a good description of the net cash flow. The Cash-Flow-Curve is negative (cash outflow more than inflow) in the beginning, then becomes positive (as sales pick up) to eventually stabilize (as the product becomes mature). The variation of funding opportunities for startups as a function of time also follows the J-Curve (Carayannis, 2013). The Cumulative-Cash-Flow diagram for a new venture, as a function of time, also follows the J-Curve (Hamermesh et al., 2002; Fig. 1). Meyer and Mathonet (2011) studied the

Internal Rate of Return (IRR) of Private Equity (PE) funds' investments over a 10-year window and showed that a) IRR follows the J-Curve, b) the IRR reaches a minimum of -10% before moving upwards, c) IRR reaches the minimum between the 1st and 2nd year, e) IIR becomes positive between the 3rd and 4th year, f) IRR plateaus and saturated at +10% over time.

From the various examples and case studies mentioned above, it is evident that J-Curve manifests itself across various disciplines (like finance, health, startups, etc.) wherever there is a "cause-and-effect" phenomena taking place in a system.

Hence, the importance of the importance of the S-Curve, as well as the J-Curve, cannot be over emphasised.

2.4 Mathematical Aspects of the J-Curve until date

The historical evolution of the mathematical aspects of the **J-Curve** (Cumulative-Growth-Curve), are now presented.

In 1983, Krueger (cited in Bahmani-Oskooee, 1985)) had given an equation of Trade Balance in terms of Exchange Rate, Domestic Price Level and Output. Bahmani-Oskooee (1985) extended this by including World Income and Domestic Money and demonstrated there was a J-Curve effect, in the study of four countries. Demirden and Pastine (1995) gave an n^{th} order auto-regression equation to show that J-Curve effect is observed in Trade Balance over time for flexible exchange rates. Bahmani-Oskooee and Fariditavana (2015) show that J-Curve effect is observed when one uses Nonlinear Auto-Regressive Distributed Lag models to study Trade Balance equations between countries. Wagner (1996) gave a simple model for the Economic Progress as a function of Exchange Rate and gave four algebraic equations in terms of Aggregate Demand, Liquidity Money, Output In terms of a time derivative) and Interest-Rates – and observed the J-Curve in a country's transformation process. Brynjolfsson *et al.* (2021) show (in terms of Regression Models) that Productivity and Output of General Purpose Technologies (e.g. Artificial Intelligence) as a function of intangible investments show a Productivity-J-Curve.

2.5 Common Platform for S-Curve and J-Curve

There is an interesting relationship between S-Curve and J-Curve. Initially, any system experiences negative returns/responses for any trigger (which corresponds to the dip in the J-Curve e.g. Cash-Flow-Curve). The cumulative sales also stays at zero (the initial flat zone in the S-Curve).

As the cash flow slowly starts picking up, the cumulative sales also picks up. When the cash flow goes positive and stabilizes, the cumulative sales attains a plateau. Hence, there seems to be a clear relation between the J-Curve and its corresponding S-Curve – see Fig. 2.5.

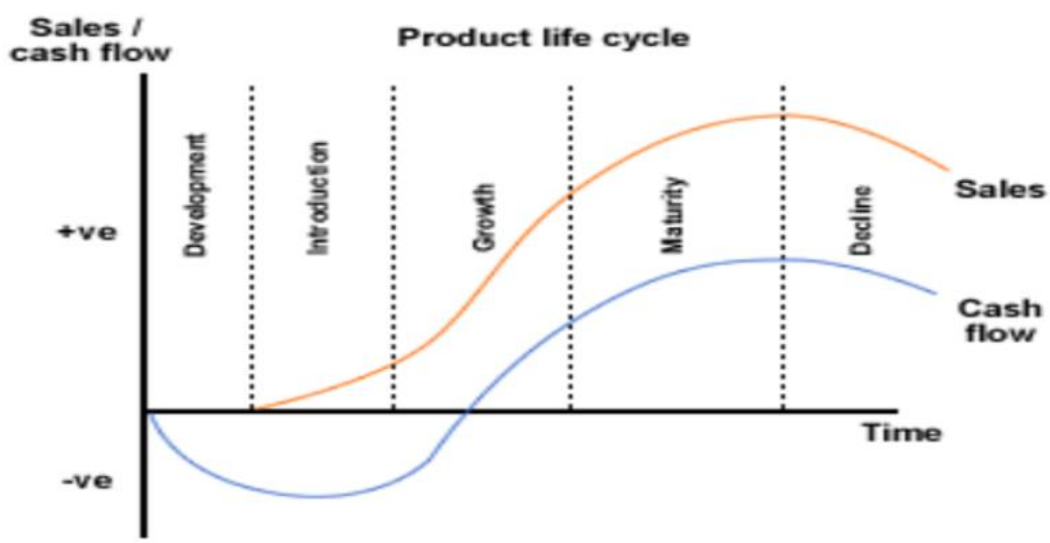


Figure 2.5 *Relation between S-Curve and J-Curve ("The product life cycle & cash flow", n.d.)*

Given this interesting connection between the S-Curve and the J-Curve, a mathematical formalism will be proposed which will put the equations describing the S-Curve and the J-Curve on the same mathematical framework.

2.6 Summary

Thus, it is observed that formal analytical studies of the S-Curve and J-Curve do not exist.

There exists a differential equation for the S-Curve, but it has not been interpreted or analysed in terms of the various ecosystem parameters. The form of the differential equation has not been used to give a physical interpretation of the underlying phenomenon.

There exists no mathematical basis for the J-Curve. Hence it is very difficult to understand and explain the following - i) what is the underlying theory of the J-Curve, ii) what are the environment parameters which determine such a behavior, iii) how can one analyse a given phenomenon and quantitatively to decide as to whether a system is displaying J-Curve or not, iv) the quantitative statistical measure to give the degree of closeness to the J-Curve (in other words – what is the amount of certainty one has to claim that a J-Curve is observed).

It is the aim of this thesis to give a formal mathematical foundation to the J-Curve, and show that it can be described by a Riccati differential equation.

The S-Curve too will be shown to be described by a Riccati differential equation – thus unifying both these curves on a uniform mathematical platform.

The solution to the J-Curve will be given in terms of a polynomial equation with suitable coefficients.

The mathematical validation of any J-Curve will be quantitatively carried out by fitting any observed data to a polynomial curve and see if the coefficients follow the desired pattern. The R-Squared value will show how closely the observed phenomenon follows the J-Curve.

We will test this mathematical theory with respect to the following case studies:

1. India's growth rate in GDP, GNI, Trade Balance and Manufacturing from 1961-2020. We expect the J-Curve in the 1991-2005 period – post-economic liberalization – and that is mathematically validated. The data of the other years shows a very poor fit to the J-Curve, as expected.
2. IRR on venture investments, and show that that indeed the mathematical conditions for the J-Curve.
3. The long-term returns from India's stock market, based on the NIFTY 500 index of the National Stock Exchange also satisfy the J-Curve conditions over the 1-2 year investment period, as expected.
4. The GDP data of various countries/regions, post the 2007-2008 financial crisis, are studied. J-Curve behaviour is observed in some cases.
5. GDP data of Croatia is analysed for potential J-Curve behaviour for two cases: post-independence (1996-2007) and post 2007-08 financial crisis. As expected, both periods show mathematically validated J-Curve characteristic.
6. While the J-Curve behaviour for response to medication could not be mathematically verified due to lack of data, the force term in the Riccati equation for the J-Curve is shown to closely follow the pharmacokinetics of drug absorption in the body.

CHAPTER III:

RESEARCH METHODOLOGY

3.1 Overview of the Research Problem

We discussed earlier that the S-Curve and J-Curve are two extremely important concepts in various fields – sociology, economics, finance, business, healthcare, startups and entrepreneurship, etc. The S-Curve is observed when studying the cumulative growth in any system (e.g. population, return on investment, profit, etc.). The J-Curve is observed in various contexts and situations which have a manifestation of Cause-and-Effect (e.g. currency depreciation-trade imbalance, medication-recovery, investment-returns, etc.).

A formal mathematical foundation and platform has been lacking till now (especially for the J-Curve). It is important to give such a formal framework for these important concepts, as it will enable a more systematic and quantitative study in all those domains where the J-Curve is manifest.

We will adopt a systematic approach to find the most suitable mathematical platform, and also to find the most natural equation which can describe the S-Curve and the J-Curve. We will show that differential equations are the natural language in this context, and that the Riccati differential equation gives a mathematical description for both the S-Curve as well as the J-Curve.

The differential equation for the S-Curve can be solved easily and the solution is shown to be the Sigmoid function. This differential equation for the S-Curve can also be recast as a nonlinear 1st order Riccati differential equation.

This suggests the following school of thought:

a) Can the J-Curve be also described in terms of the differential equation?

b) Can this also be in the form of a nonlinear 1st order Riccati differential equation?

c) Just as the Sigmoid function is the closed form solution of the S-Curve, can a closed form solution for the J-Curve also be found?

d) Can specific criteria/conditions be specified which can mathematically validate the existence of the J-Curve?

But finding the corresponding Riccati differential equation, as well as the solution, for the J-curve solution is not easy – formal mathematical analysis, as well as physical reasoning will be used to ultimately find them.

We will show that the J-Curve can indeed be described by a nonlinear 1^{st} order Riccati differential equation, and its associated linear 2^{nd} order differential equation – the Laguerre differential equation. The solution can also be mathematically described in terms of polynomials similar to the Laguerre polynomials (one of the extremely important class of polynomials in science and engineering).

Once the mathematical foundation and equation for the J-Curve have been found, they will be tested with respect to various economic and financial case studies described in the previous chapter. These data will confirm that the J-Curve equation proposed does indeed explain and mathematically validate the J-Curve phenomena.

The functional form of the Riccati equation for the J-Curve will also be used to throw new light on the pharmacokinetics of medicine absorption in a body.

3.2 Research Hypothesis

We will set up a mathematical foundation and formulation of the S-Curve and the J-Curve via the following results steps and procedures.

H1: The differential equation for the S-Curve can be recast as a Riccati equation.

H2: The J-Curve can be described by a differential equation, which is also in the form of the Riccati equation.

H3: A common mathematical formalism to the S-Curve, as well as the J-Curve, can be given in the framework of a general nonlinear 1^{st} order Riccati equation.

H4: A suitable linear 2^{nd} order differential can be found to describe the J-Curve in terms of the Laguerre polynomials. This gives the functional form for the J-Curve.

H5: A quantitative process can be set up to mathematically verify any curve that will be classified as a J-curve.

H6: Cases in which the J-curve is expected / observed in practice can be mathematically proven to show the behavior of the J-curve.

H7: Functional form of medicine absorption in the body will be presented.

3.3 Research Approach Flow Diagram

The essential process flow described above is captured in Table 3.1 below.

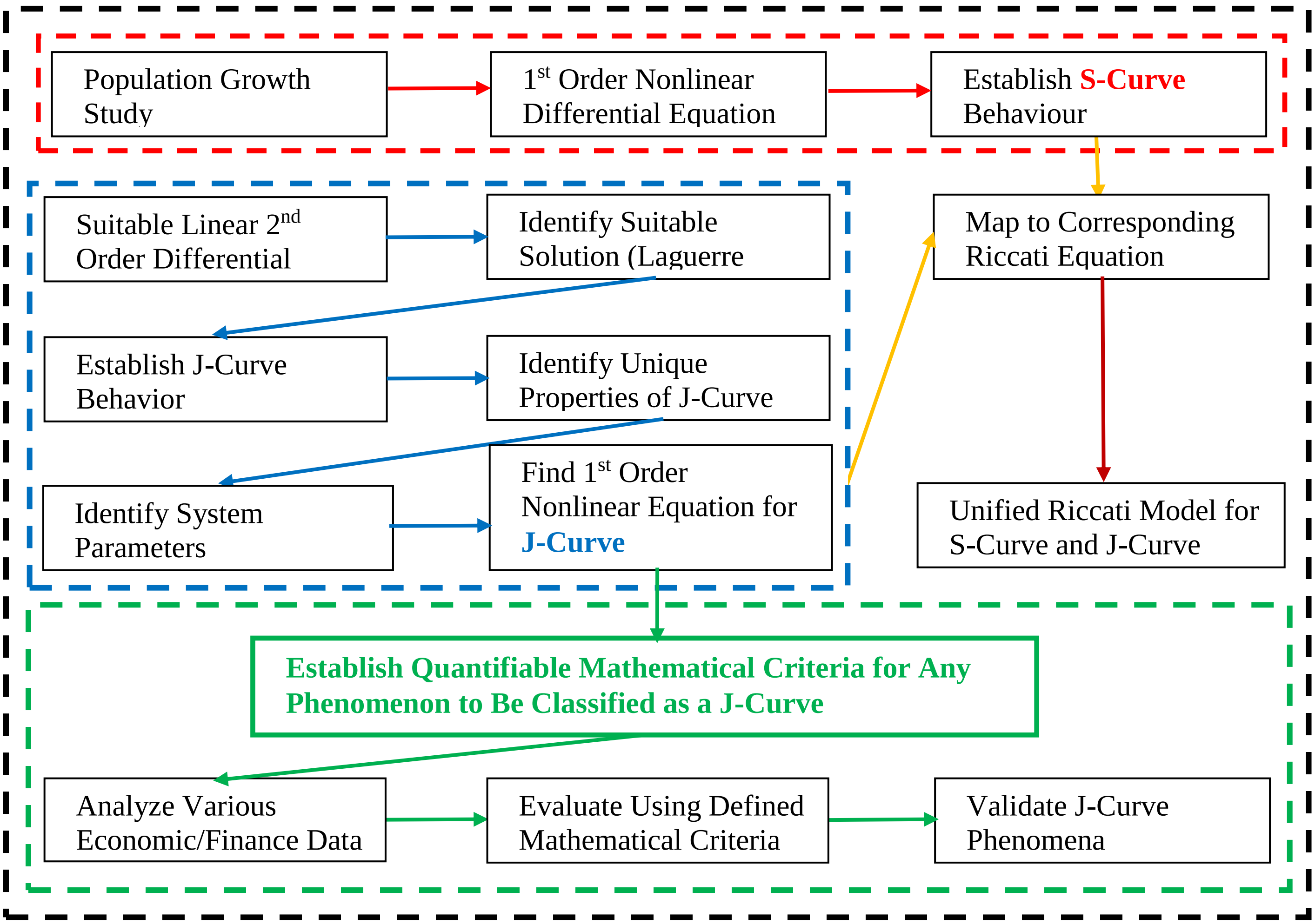


Figure 3.1 *Research Methodology Flow Diagram*

3.4 Expected Outcome of the Research

The following new results and findings are expected to come out of this research:

We will be show for the **first time** that:

1. An important differential equation – the Riccati equation – can be used used as a common platform to give a mathematical basis for the S-Curve, as well as the J-Curve.
2. The J-Curve is governed by differential equation, just like the S-Curve.
3. The J-Curve can be described in terms of a nonlinear 1st order Riccati differential equation, as well as its associated Laguerre Polynomial (an important orthogonal polynomial) via the linear 2nd order Laguerre differential equation.
4. The equation for the J-Curve can be described by a polynomial equation with coefficients alternating in sign (similar to the Laguerre polynomials).
5. It is possible to establish quantifiable and mathematical criteria for any phenomenon to be classified as a J-Curve.
6. One can interprete the role of ecosystem parameters involving startups, competition, customers and markets, environmental aspects financing, etc. in the mathematical model.
7. Ecosystem parameters determine the evolution of the J-Curve (eg. Cash-Flow-Curve in most startups).
8. Various disparate disciplines like finance, medicine and health, startups, etc. which manifest J-Curves, can all now be quantitatively studied with precise mathematical equations – and validated if they are indeed characterized by a J-Curve.

CHAPTER IV:

MATHEMATICAL MODELLING OF THE S-CURVE AND J-CURVE

4.1 Modeling the S-Curve

We have discussed earlier that in 1838 by Verhulst gave the differential equation for the S-Curve (see eqn. (2.1)) which can be re-written in the form

$$\frac{dP(t)}{dt} - A * P(t) - A * \frac{P(t)^2}{B} = 0 \qquad 4.1$$

This differential equation for the population growth can be thought of as a special case of the general Riccati equation (eqn. (A.1) in Appendix A)

$$\frac{dy(t)}{dt} + Q(t) * y(t) + R(t) * y(t)^2 = P(t) \qquad 4.2$$

Where Q = -A, R = A/B, and P = 0. The solution to eqn. (4.1) can be shown to be (see eqn. (2.2))

$$P(t) = \frac{[P(0) * e^{At}]}{\left[1 + P(0) * \frac{e^{At} - 1}{B}\right]} \qquad 4.3$$

This is the Logistics-Curve, a special case of the Sigmoid function (Wood, 2020), and is the typical S-Curve (see Fig. (2.2)).

Thus, we see that the <u>differential equation for the S-Curve is a special case of the Riccati differential equation</u>. Below we shall explore setting up the J-Curve as another special case of the Riccati differential equation.

4.2 Modeling the J-Curve

As per the research methodology flow diagram in chapter 3 (Fig. (3.1)), we start with looking for a suitable linear 2nd order differential equation. As described in detail in Appendix B, 2nd order linear ordinary differential equations play a very important many areas of engineering (e.g. fluid mechanics, solid mechanics, telecommunications, heat transfer, etc.) and sciences (e.g. quantum mechanics, electromagnetic theory, classical mechanics, etc.).

The reason for looking at these equations is to explore potential candidates which can provide a mathematical formulation of the J-Curve. If we can find a suitable candidate, then the result of Appendix A can be used to find the corresponding Riccati equation and vice-versa. This, in turn, will lead us to understand how the ecosystem can play a role in a system following the J-Curve.

<u>We propose the equation for the J-Curve</u> to be the 1st order Riccati equation

$$t * \frac{dy(t)}{dt} + \mathrm{e}^{\mathrm{t}} * y(t)^2 = - * t * e^{-t} \qquad 4.4$$

Since this equation is difficult to solve, we convert it into its associated 2nd order differential equation (see Appendix A)

$$t * \frac{d^2u(t)}{dt^2} + (1 - t) * \frac{du(t)}{dt} + n * u(t) = 0 \qquad 4.5$$

This is the Laguerre equation whose solutions are the famous Laguerre polynomials (see Appendix C), and which have the form of a J-Curve.

As per the analysis in Appendix B, the hypergeometric differential equations are good candidates to consider – and further physical conditions of satisfying the "Cause-and-Effect" property (which is typical of the J-Curve) gives us the equation for the Laguerre polynomial as the most promising candidate.

This very important Riccati differential equation, eqn. (4.4), is proposed as the equation to describe the J-Curve, and the equations describing the J-Curve are polynomials similar to the Laguerre polynomials, where the polynomials having coefficients of alternating signs, are proposed as equations which describe the J-Curve (discussed in detail in Appendix C).

4.3 Interpretation of the Riccati Equation of the J-Curve

The Riccati equation for the J-Curve, eqn. (4.4) is a particular form of the generic equation mention in Appendix D (see eqn. D.4).

$$I(t) * \frac{dy(t)}{dt} + D(t) * y(t)^2 = F(t) \qquad 4.6$$

Here a) I(t) is the time-dependent inertia of the system under study (similar to the mass for the example of the particle), b) the damping parameter on the system evolving in an ecosystem/environment which produces a drag is characterized by the damping parameter D(t) (similar to the viscous damping on the particle), c) the system evolves under the influence of external environmental/ecosystem forces F(t) (similar to the gravitational force on the particle). The inertia I(t) is the likely reason for the lag observed in the Cause-and-Effect scenario - e.g. change in trade imbalance due to currency depreciation.

Comparing with the J-Curve equation (4.4) with eqn. (4.6), the following parameters are identified for any system/phenomenon displaying the J-Curve characteristic.

The inertia (analogous to mass) grows linearly with time.

$$I(t) = t \qquad 4.7$$

The drag of the environment grows exponentially with time.

$$D(t) = e^t \quad 4.8$$

The force due to the external ecosystem gives an initial and positive linear impetus, but later exponentially fades away to zero.

$$F(t) = t * e^{-t} \quad 4.9$$

We propose that all systems which display the J-Curve have their ecosystem parameters, which vary in time as given in eqns. 4.7, 4.8 and 4.9.

4.4 Ecosystem Factors Contributing to I(t), D(t) and F(t) in Riccati Equation of the J-Curve

For any general financial system, we attribute various factors can be associated with the Inertia I(t), Damping D(t) and External-Force F(t). Some of these are given below in Table 4.1.

I(t): Inertia = Systems Attributes	**D(t): Damping = Ecosystem Limitations**	**F(t): Force = External Influence**
. Planning and strategy . Type of policies . Domain knowledge . Size and complexity . Infrastructure and resources available . Experience available	. Competition . Cost of implementation . Limitations to scale . Lack of acceptability by stakeholders . Lack of resources . Existing barriers	. New opportunities . Policies and laws . Collaborators and partners . Resources and funding . Technology . Acceptability

Table 4.1 *Physical Parameters Impacting the J-Curve for Financial Systems*

In healthcare, we associate the below factors with the Inertia I(t), Damping D(t) and External-Force F(t). Some of these are given below in Table 4.2

I(t): Inertia = Patient's History	**D(t): Damping = Health Limitations**	**F(t): Force = External Influence**
. Weight . Age . Health history .	. Illness history . Co-morbidities .	. Type of medication . Efficacy of medication . Pharmaco-kinetics/dynamics of medication

Table 4.2 *Patient's Parameters Impacting the J-Curve for Medical Treatment*

4.5 Physical Interpretation of the J-Curve Differential Equation

The differential equation for the J-Curve is given in eqn. (4.4). This describes any system dynamics, which manifests a J-Curve behavior. Analogous to the Riccati equation for a particle moving under an external gravitation force in a viscous/damping medium – we shall interpret eqn. (4.4) for a system moving under an external ecosystem force in an environment, which causes a drag on the system.

Eqns. (4.7 – 4.9) give the explicit forms of the inertia I(t), damping D(t) and force F(t). For any system to exhibit the J-Curve, it is postulated that

a) its inertia to be of the form $I(t) = t$,
b) the damping effect to be of the form $D(t) = e^{t}$,
c) External influence to be of the form $F(t) = t * e^{-t}$.

The external influence due to the ecosystem has an interesting behavior – it grows linearly in the beginning, then decays exponentially to zero. Its graph is represented (where the X-axis is in arbitrary units) as

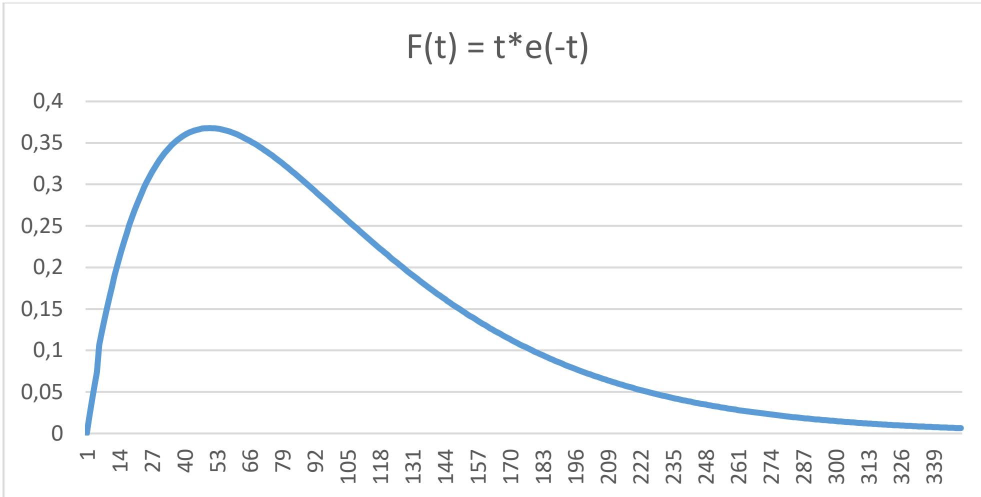


Figure 4.1 $F(t) = t * e^{-t}$

From the time F(t) starts acting on the system till it almost reaches zero – it reaches its maximum at about 1/7 of the total duration. This approximately corresponds to about 14% of the total time before the effect of F(t) drops to zero. This will be later related to the very important aspect of pharmacokinetics of medicine absorption in the body.

<u>It is postulated that the J-Curve is described by a) the nonlinear 1st order Riccati equation, b) its associated linear 2nd order Laguerre differential equation, c) the functional form of the J-Curve are similar to the Laguerre polynomials</u>. (Usually, the order of the polynomial depends on the number of inflexions – if the curve has N inflexions, then it is described by the (N+I)th Laguerre polynomial L_{N+1}.

Any system manifesting the J-Curve - IRR from an investment, or response/effect of a system to a cause - which initially drops, then picks up and stabilizes has two points of inflexion (see Appendix C which describes properties of the Laguerre polynomials). Hence, in this example, the typical J-Curve can be represented by the 3rd order Laguerre polynomial L_3, which is plotted below. It is interesting to note that minimum of L_3 occurs at the 25% mark.

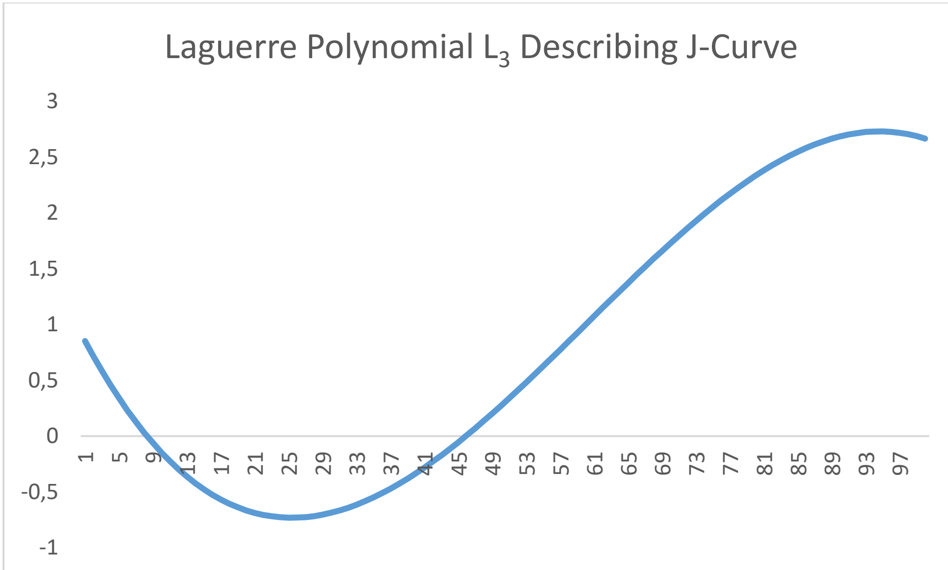


Figure 4.2 *Laguerre Polynomial L_3*

Combining the 14% maxima of external force F(t) (aka the Cause) with the L_3 solution for the J-Curve (aka the Effect) which has a minima at 25% - it can be seen that L_3 continues to drop and lags behind F(t) even though it initially increases – a very typical characteristic of the J-Curve showing the Cause-and-Effect behavior.

The interpretation of the Riccati differential equation (eqn. (4.4)) in the context of healthcare and pharmacokinetics will be given in Appendix E.

4.6 Key Results

We have proposed the differential equation to describe any system manifesting the J-Curve in eqn. (4.4) as a nonlinear 1st order Riccati equation, along with its linear 2nd order Laguerre differential equation (eqn. (4.5)).

The mathematical formulation for the J-Curve in the previous chapter had provided clear criteria for any system to manifest J-Curve phenomenon:

1. There has to be a clear “Cause-and-Effect” relationship where a certain cause results in a time-lagged effect.
2. The perceived curve to have a nonlinear polynomial curve fitted.
3. The polynomial equation should have coefficients of alternating sign – this is a clear property of the Laguerre polynomials. If this condition is violated, one can conclude that there is no manifest J-Curve phenomenon.
4. Based on the context, a suitable R-squared value to be considered as a secondary filter to decide if a system is exhibiting J-Curve behavior.
5. If the above polynomial condition is satisfied, then one should look at the behavior of other variables/parameters which could also likely be affected by the cause. If those variables also satisfy the above conditions, it is safe to conclude that there is a manifest J-Curve phenomenon being manifest.

Two important mathematical criteria are proposed for any curve to be mathematically classified as a J-Curve:

C1: Any system or phenomenon which claims to follow the J-Curve should have the curve fitted by a polynomial equation, with coefficients alternating in sign (similar to the Laguerre polynomials.

C2: The R-squared value of the curve fitting is required to be greater than or equal to 0.6. (It is recognized that this is a very subjective condition, this threshold will likely vary from context to context).

In the following chapters, the following will be analysed form the mathematical aspects of the J-Curve:

A) Economic data for India will be studied from 1961 to 2020 will be analysed to see if a) any J-Curve is observed over specified time intervals, b) if the J-Curve behaviour exhibited does indeed satisfy the two conditions outlined above, c) whether the observed J-Curves (if any) can be explained in terms of any Cause-and-Effect relationship with respect to major changes in economic policies of the nation, d) if reasonable conclusions on the J-Curve behaviour can be formed by evaluating the R-squared values.

B) Data from simulation studies carried out regarding the return on investments in startups will be analysed to see if they mathematically satisfy the criteria of J-Curve.

C) Data for return-on-investments in shares market will be studied to see if the returns follow the expected trend and, in addition, flow a J-Curve.

D) The GDP data of some countries/regions will be studied to see their behaviour post the 2007-08 financial crisis. As expected, quite a few show J-Curve behaviour.

E) The GDP of Croatia is analysed following the post-independence period, and also the post financial crisis of 2007-08. Expectedly, they show J-Curve characteristics.

F) Some interesting aspects of the Riccati equation for the J-Curve will be highlighted in the context of absorption of medicine in the body in the treatment of patients for various illnesses.

Each of the above will be studied and analysed based on the mathematical model set up in this thesis.

The process described above is summarized in the below Fig. 4.3:

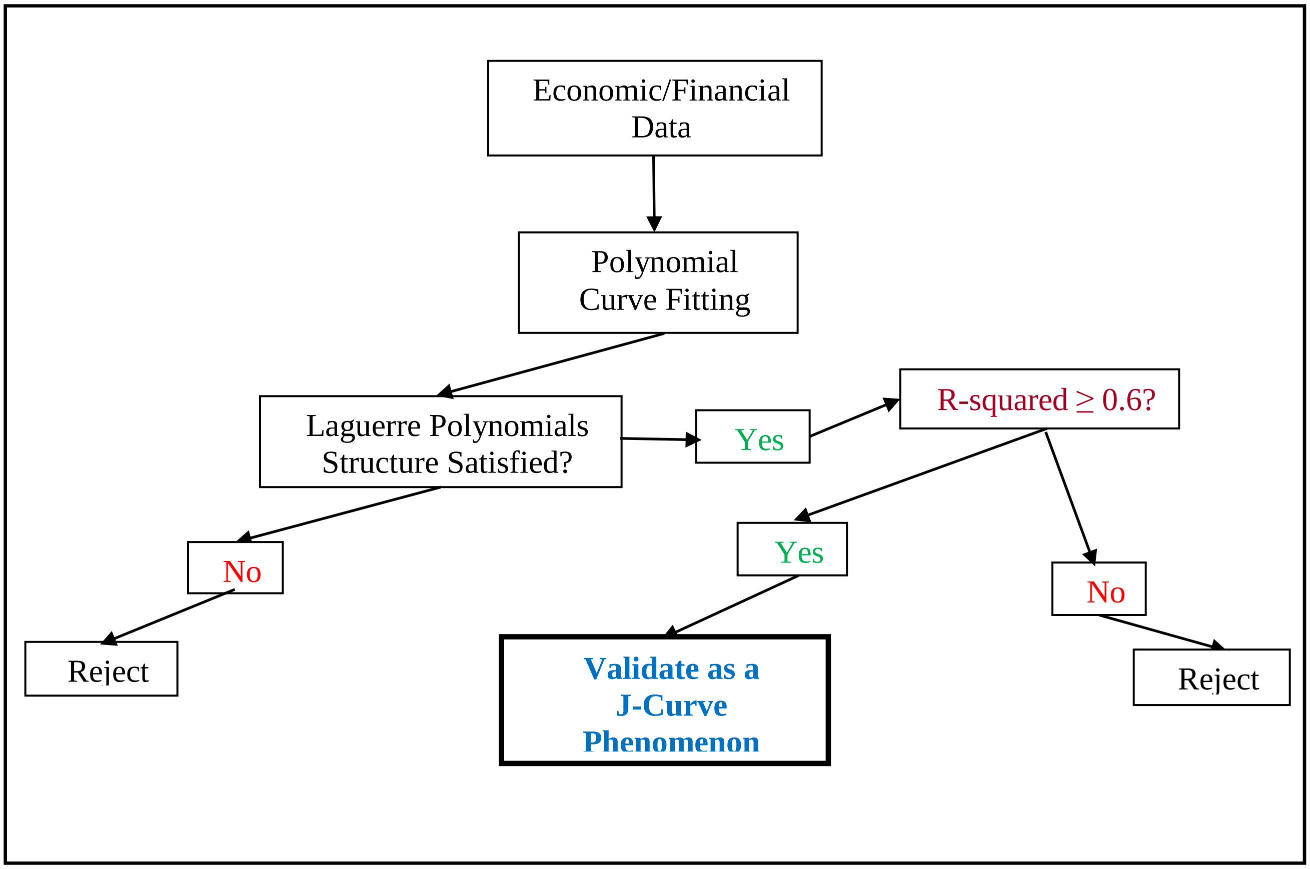


Figure 4.3 *Process for J-Curve Validation*

CHAPTER V:

CASE STUDY 1: J-CURVE IN INDIA'S ECONOMIC DATA 1961-2020

5.1 Data Analysis and Relation to the Mathematical Formalism of J-Curve

We presented the mathematical formalism for the J-Curve in the previous chapter in terms of either a) nonlinear 1^{st} order Riccati equation (eqn. (4.4)) or, b) linear 2^{nd} order Laguerre differential equation (eqn. (4.5)).

The ecosystem parameters, and their role in determining the dynamics of the J-Curve via the inertia I(t), damping coefficient D(t) and the external forces/influence F(t), were presented in Table 4.1(for financial systems) and in Table 4.2 (for medical applications).

We apply this theory to real case data to see if the mathematical formulation help us to quantitatively predict J-Curve behavior.

The important case study analyzed now will be that of the economic data of India in the period 1961-2020 in terms of below four important parameters:

a) India Trade Balance (National Bureau of Economic Research, 2021)
b) India GDP (Gross Domestic Product) Growth Rate ("India GDP 1960-2021", n.d.),
c) India GNI (Gross National Income) Growth Rate ("India GNI 1962-2021", n.d.)
d) India Manufacturing Growth Rate ("India Manufacturing Output 1960-2022", n.d.)

These four parameters have been chosen since they reflect macroeconomic metrics of any nation, and are likely affected by any major economic policies. Also, the growth rates are taken for each of these four parameters, to give a better understanding of year-on-year growth.

Hashim *et al.* (2011) have discussed the manifestation of J-Curve in India's economy post the 1991 liberalization. The context of J-Curve behavior, post liberalization, was also discussed in India's manufacturing sector (Virmani and Hashim, 2011). Based on the mathematical modeling for the J-Curve in the previous chapters, these phenomena will be analyzed based on the economic data to give a quantitative validation of the J-Curve in India's post liberalization economy.

Analysis of the data will be based on the nonlinear polynomial curve-fitting of the data over the above periods, using the built-in feature in MS-Excel. Since J-Curve follows the characteristics of Laguerre polynomials, nonlinear polynomial curve fitting will be done. Agreement, or disagreement, of the data with the J-Curve will be based on the following criteria:

C1. Like the Laguerre polynomials, the polynomial equations of the curve fitting must have coefficients which alternate in sign. Any equation which does not have coefficients which alternate in sign will be rejected as it will not have the property of Laguerre polynomial, will describe the J-Curve.

C2.a. If the above criteria is not satisfied, then the value of R-squared is not considered, hence not applicable (NA). The curve does not satisfy the criteria for J-Curve.

C2.b. If the first (above) criteria is satisfied, then it is required that the R-squared value should be more than 0.6. This is a subjective criteria and can/will vary from context to context. It needs to be seen if this leads to conclusions which are clear and unambiguous.

5.2 Predictions Based on Mathematical Formalism of J-Curve

India gained its independence in 1947 and started re-building the nation in terms of education, infrastructure, healthcare, political and policy platforms, etc. Many industries started growing and flourishing over the years.

In 1991, a major policy was implemented in terms of liberalization the economy (Mehra, 2021). This brought about major changes in the growth of country. Since such a change will naturally result in the major economic metric – and this is a natural case of "Cause-and-Effect", it will be interesting to see if four of the important metrics in the growth rate of – Trade Balance, GDP, GNI and Manufacturing – displayed the J-Curve effect. This would be a manifestation that the Liberalization did indeed have an effect on these economic metrics.

To ensure that there is no bias in the analysis and judgment, data for the four economic metrics will not be analyzed only for the 10-15 years starting 1991. The data will be studied for 60 years from 1961 till 2020. Since economic changes at a country level take 10-15 years to manifest, the data for these 60 years will studied in 4 stages:

i. 1961-1975,
ii. 1976-1990,
iii. 1991-2005 and
iv. 2006-2020.

Predictions based on the J-Curve Theory: Based on the mathematical formulation, it is predicted that the following will be observed by analyzing the data over 60 years in the four sub-periods.

a) There should be no J-Curve observed for the periods 1961-1975, 1976-1990 and 2006-2020, in any of the above four economic parameters, as no major economic policy changes were instituted in those periods.

b) J-Curve should be observed for all the four metrics for the period 1991-2005. The economic liberalization started in 1991, and took some years for the effect to be observed at the national economic level.

c) If all the four parameters/metrics show J-Curve behavior, as per the mathematical guidelines stipulated – it would be fair to conclude that a mathematical justification and validation has been given for the predicted J-Curve.

5.3 India Trade Balance Growth Rate as % of GDP 1961-2020

	% of GDP	Year	% of GDP	Year	% of GDP	Year	% of GDP
1961	-1.65%	1976	0.57%	1991	0.00%	2006	-3.19%
1962	-1.86%	1977	0.12%	1992	-0.75%	2007	-4.09%
1963	-1.63%	1978	-0.27%	1993	0.02%	2008	-5.17%
1964	-1.96%	1979	-1.42%	1994	-0.30%	2009	-5.47%
1965	-1.90%	1980	-3.11%	1995	-1.18%	2010	-4.45%
1966	-2.53%	1981	-2.64%	1996	-1.16%	2011	-6.54%
1967	-1.91%	1982	-2.16%	1997	-1.24%	2012	-6.72%
1968	-0.90%	1983	-2.02%	1998	-1.66%	2013	-2.98%
1969	-0.32%	1984	-1.44%	1999	-1.91%	2014	-2.99%
1970	-0.10%	1985	-2.39%	2000	-0.91%	2015	-2.30%
1971	-0.34%	1986	-1.83%	2001	-0.88%	2016	-1.77%
1972	0.32%	1987	-1.38%	2002	-0.98%	2017	-3.16%
1973	-0.51%	1988	-1.42%	2003	-0.70%	2018	-3.72%
1974	-1.19%	1989	-1.14%	2004	-1.79%	2019	-2.53%
1975	-1.00%	1990	-1.40%	2005	-2.79%	2020	-0.32%

Table 5.1.a *Patient’s Parameters Impacting the J-Curve for Medical Treatment*

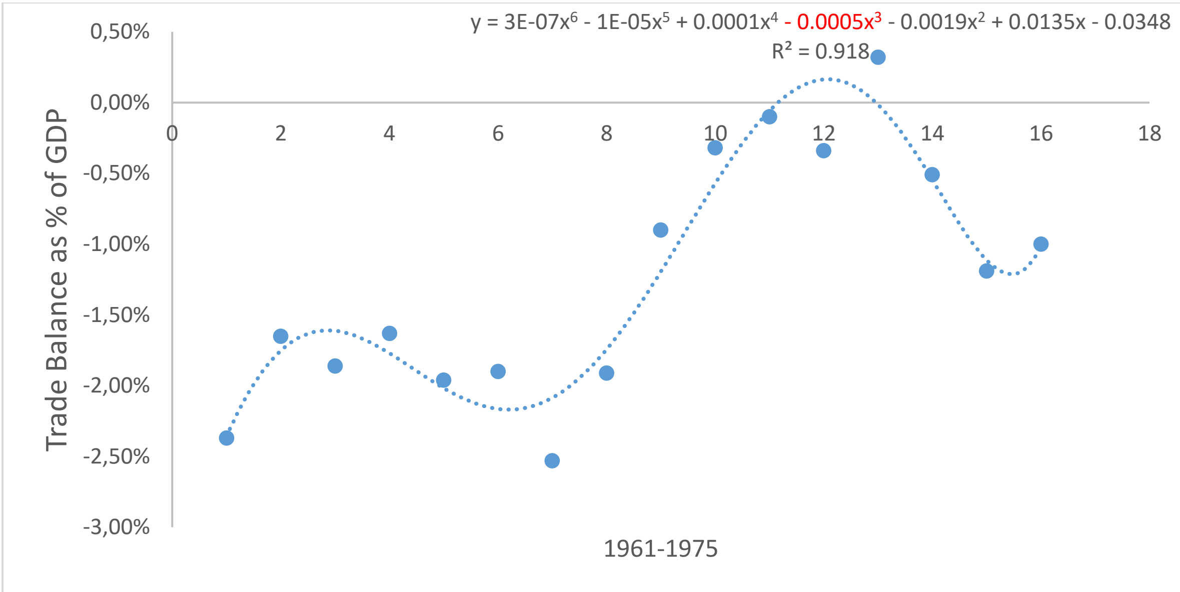


Figure 5.1.a *India Trade Balance Growth Rate 1961-1975*

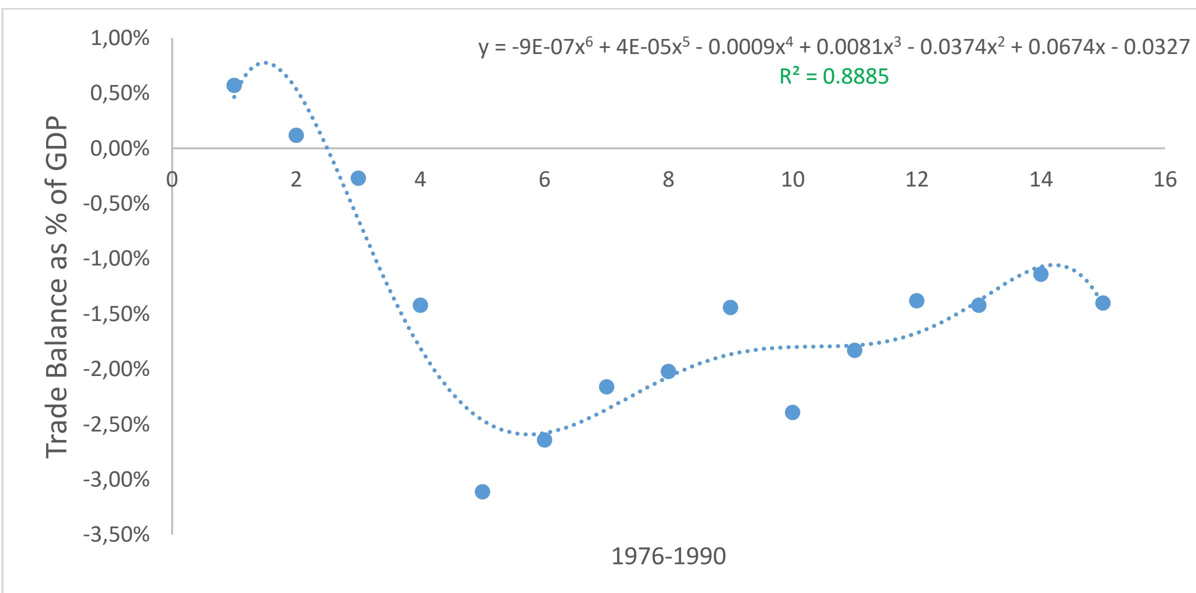


Figure 5.1.b *India Trade Balance Growth Rate 1976-1990*

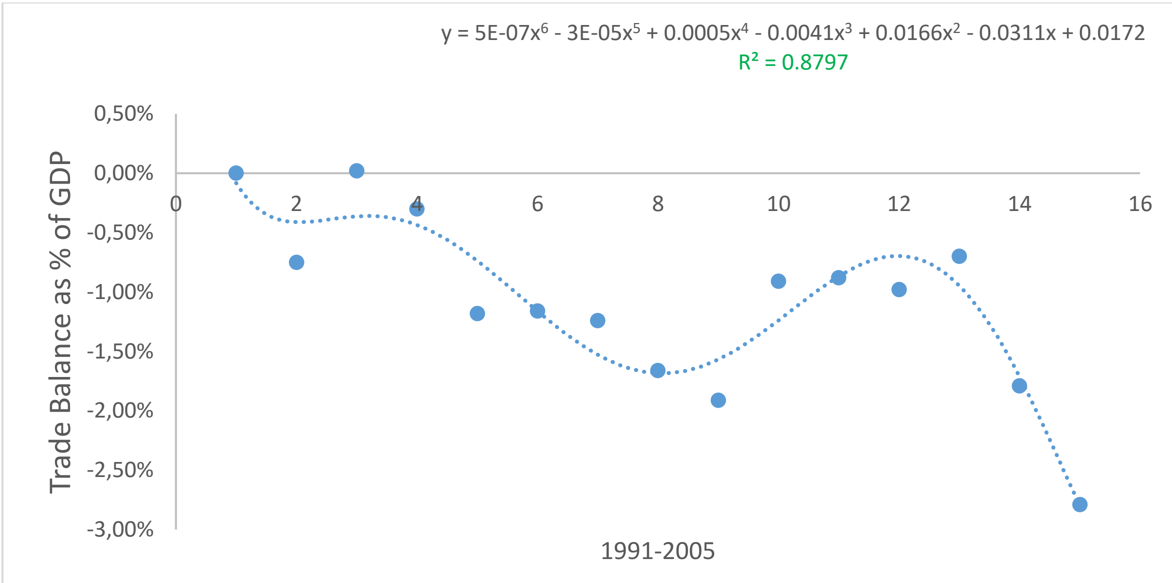


Figure 5.1.c *India Trade Balance Growth Rate 1991-2005*

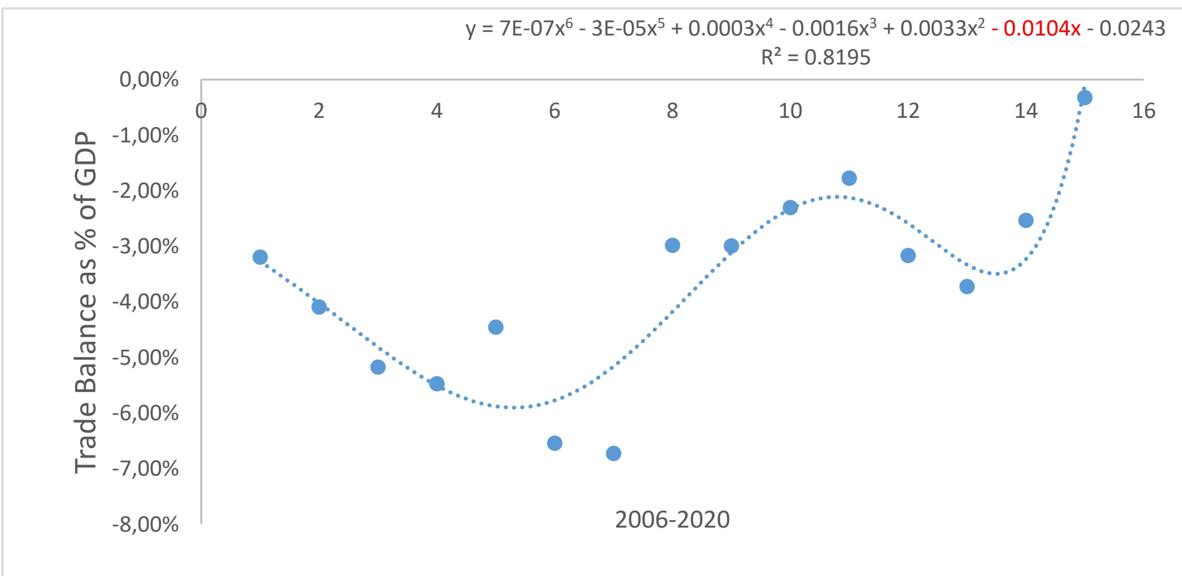


Figure 5.1.d *India Trade Balance Growth Rate 2006-2020*

The results of the above nonlinear curve fittings, based on the criteria explained in section 5.1, are summarized in Table 5.1.b below (**Trade Balance Growth Rate):**

Period	1961-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	No	Yes	Yes	No
R-Squared Criteria	NA	Yes (.8885)	Yes (.8797)	NA

Table 5.1.b *India Economic Growth Rate 1961-2020*

5.4 India GDP Growth Rate Data 1961-2020

Year	Growth	Year	Growth	Year	Growth	Year	Growth
1961	3.72%	1976	1.66%	1991	1.06%	2006	8.06%
1962	2.93%	1977	7.25%	1992	5.48%	2007	7.66%
1963	5.99%	1978	5.71%	1993	4.75%	2008	3.09%
1964	7.45%	1979	-5.24%	1994	6.66%	2009	7.86%
1965	-2.64%	1980	6.74%	1995	7.57%	2010	8.50%
1966	-0.06%	1981	6.01%	1996	7.55%	2011	5.24%
1967	7.83%	1982	3.48%	1997	4.05%	2012	5.46%
1968	3.39%	1983	7.29%	1998	6.18%	2013	6.39%
1969	6.54%	1984	3.82%	1999	8.85%	2014	7.41%
1970	5.16%	1985	5.25%	2000	3.84%	2015	8.00%
1971	1.64%	1986	4.78%	2001	4.82%	2016	8.26%
1972	-0.55%	1987	3.97%	2002	3.80%	2017	6.80%
1973	3.30%	1988	9.63%	2003	7.86%	2018	6.53%
1974	1.19%	1989	5.95%	2004	7.92%	2019	4.04%
1975	9.15%	1990	5.53%	2005	7.92%	2020	-7.96%

Table 5.2.a *India GDP Growth Rate 1961-2020*

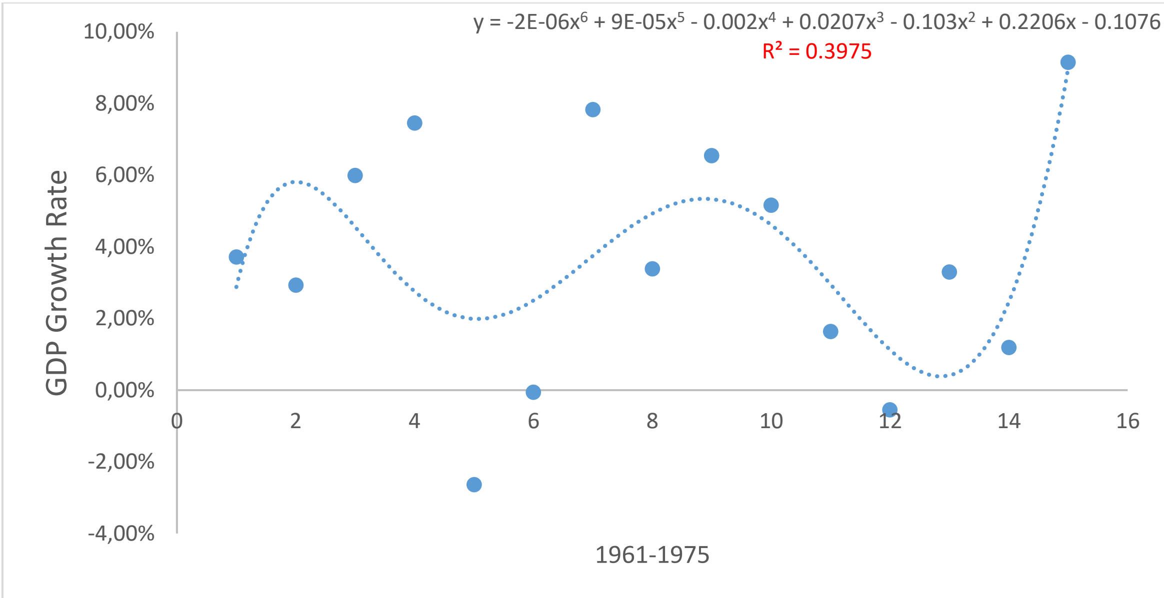


Figure 5.2.a *India GDP Growth Rate 1961-1975*

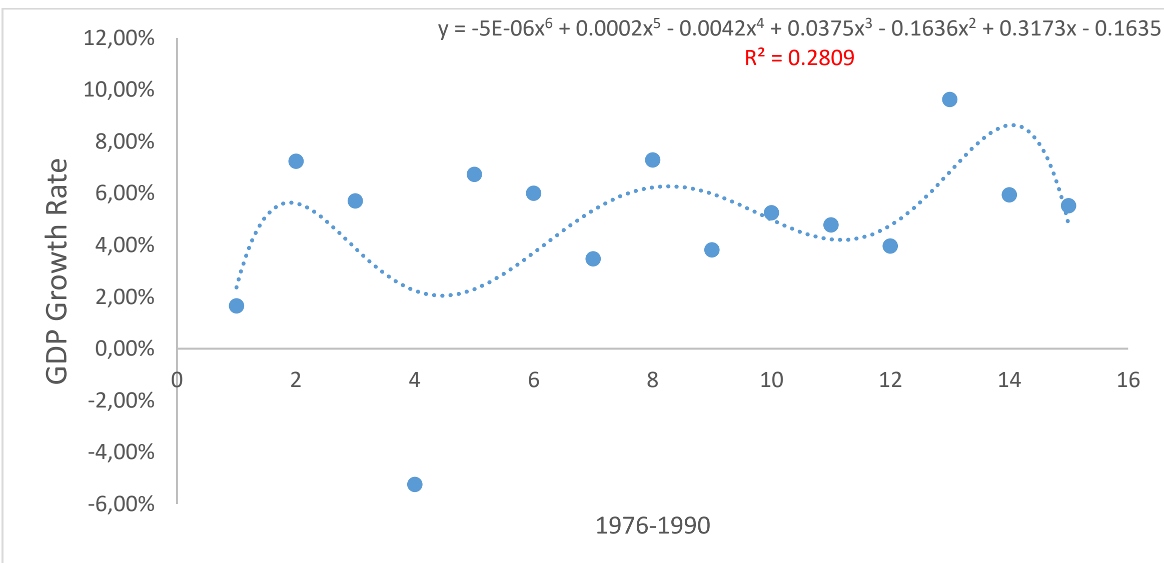


Figure 5.2.b *India GDP Growth Rate 1976-1990*

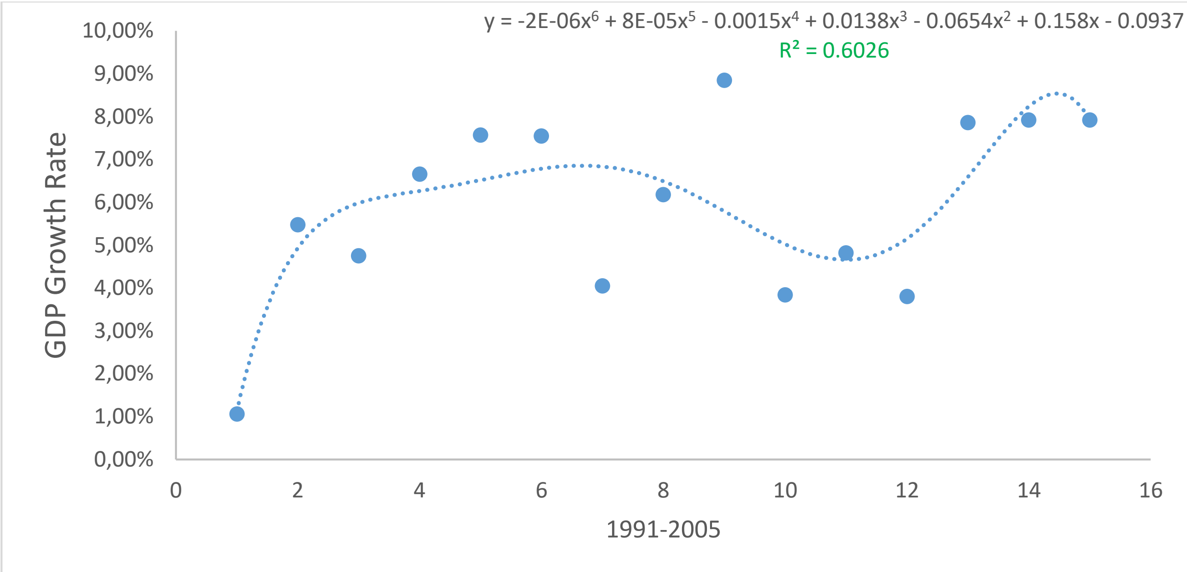


Figure 5.2.c *India GDP Growth Rate 1991-2005*

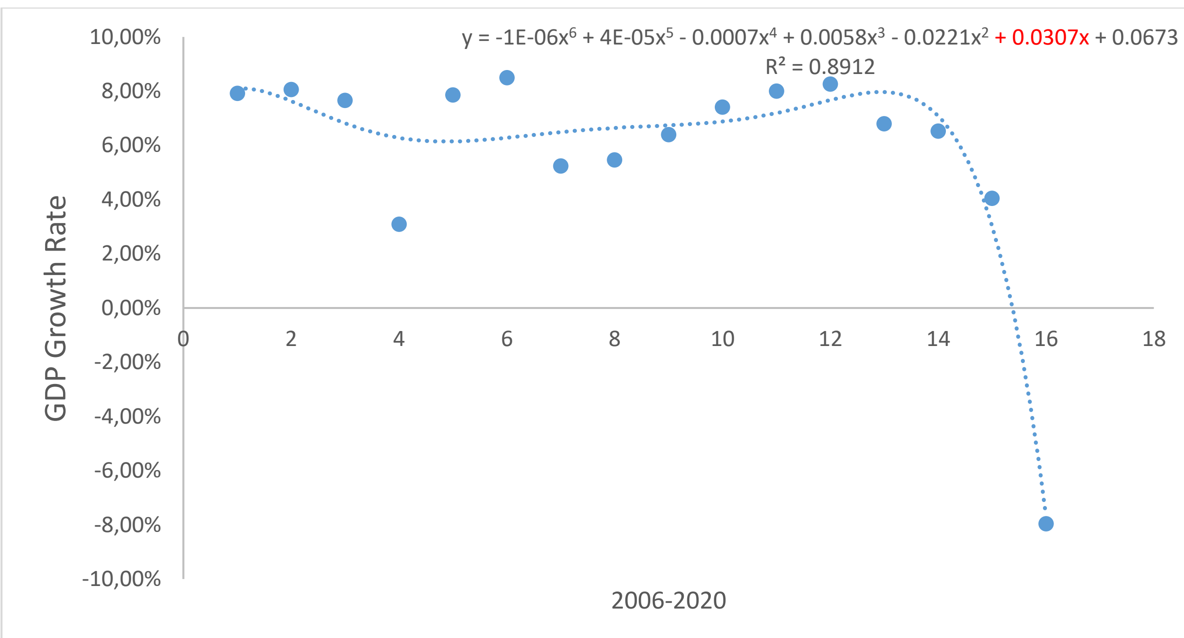


Figure 5.2.d *India GDP Growth Rate 2006-2020*

The results of the above nonlinear curve fittings, based on the criteria explained in section 5.1, are summarized in Table 5.2.b below (**GDP Growth Rate):**

Period	1961-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	Yes	Yes	Yes	No
R-Squared Criteria	No (.3975)	No (.2809)	Yes (.6026)	NA

Table 5.2.b *India GDP Growth Rate 1961-2020*

5.5 India GNI Growth Rate Data 1961-2020

Year	Growth Rate	Year	Growth Rate	Year	Growth Rate	Year	Growth Rate
1961	3.61%	1976	1.69%	1991	0.83%	2006	7.99%
1962	2.92%	1977	7.28%	1992	5.48%	2007	8.04%
1963	6.04%	1978	5.80%	1993	4.90%	2008	2.90%
1964	7.40%	1979	-5.00%	1994	6.78%	2009	7.86%
1965	-2.69%	1980	6.85%	1995	7.74%	2010	7.97%
1966	-0.19%	1981	5.79%	1996	7.74%	2011	5.45%
1967	7.88%	1982	3.13%	1997	4.13%	2012	5.14%
1968	3.41%	1983	7.19%	1998	6.19%	2013	6.31%
1969	6.57%	1984	3.68%	1999	8.94%	2014	7.49%
1970	5.18%	1985	5.32%	2000	3.56%	2015	8.02%
1971	1.67%	1986	4.72%	2001	5.02%	2016	7.30%
1972	-0.53%	1987	3.80%	2002	3.99%	2017	7.82%
1973	3.36%	1988	9.30%	2003	7.79%	2018	6.55%
1974	1.32%	1989	5.82%	2004	7.96%	2019	4.16%
1975	9.23%	1990	5.38%	2005	7.90%	2020	-7.97%

Table 5.3.a *India GNI Growth Rate 1961-2020*

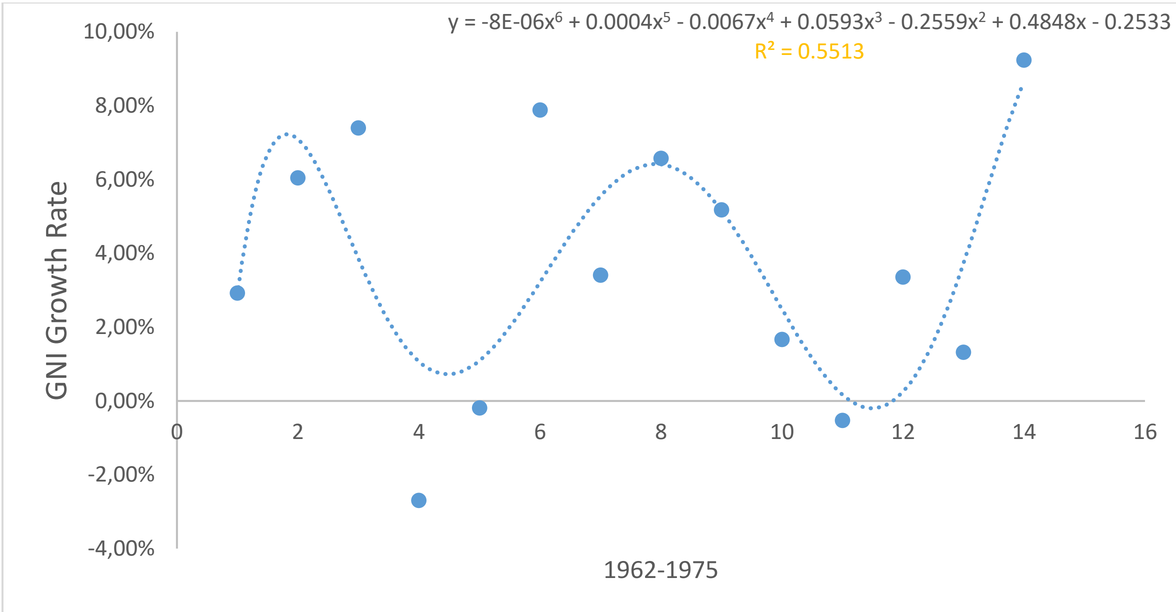


Figure 5.3.a *India GNI Growth Rate 1962-1975*

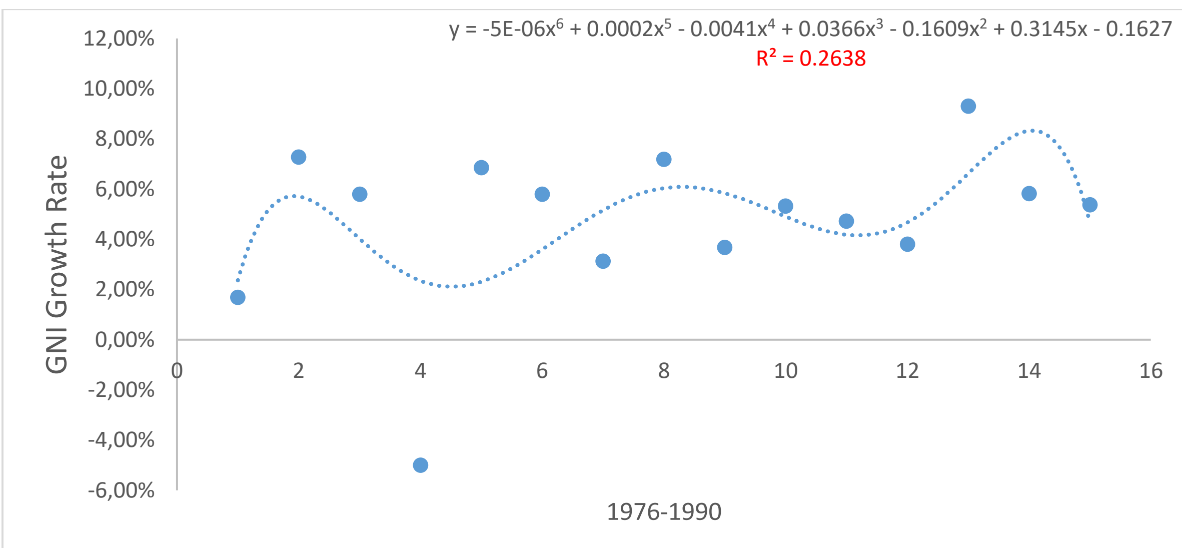


Figure 5.3.d *India GNI Growth Rate 1976-1990*

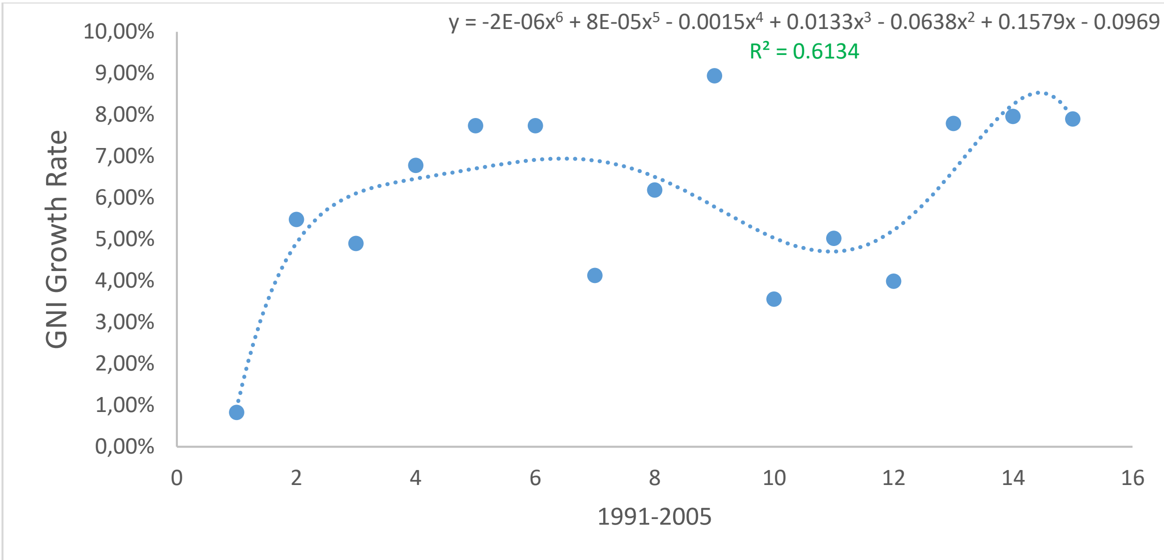


Figure 5.3.c *India GNI Growth Rate 1991-2005*

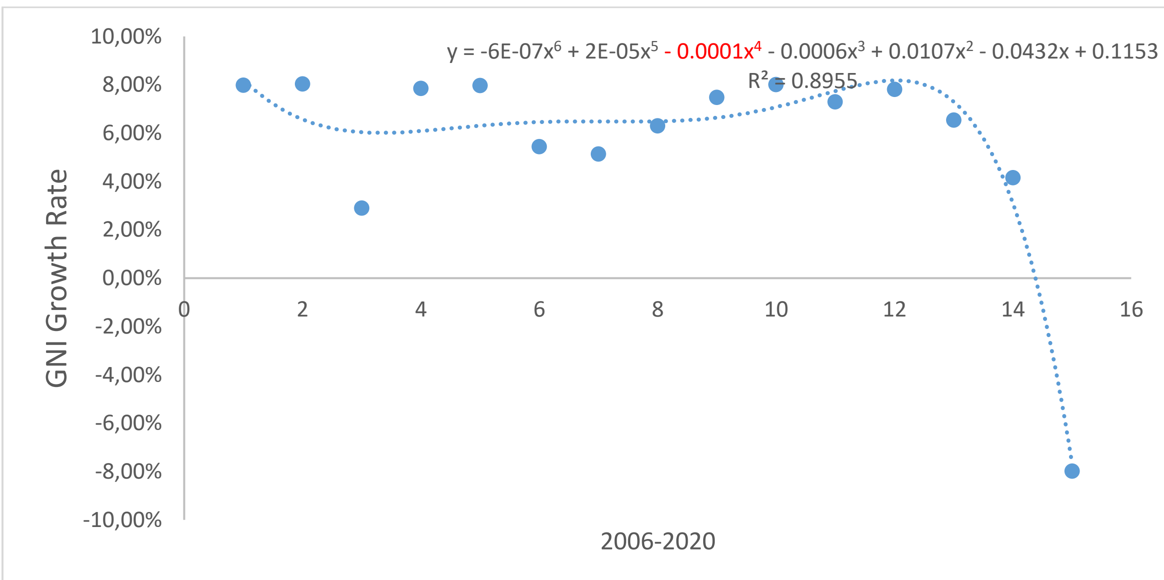


Figure 5.3.d *India GNI Growth Rate 2006-2020*

The results of the above nonlinear curve fittings, based on the criteria explained in section 5.1, are summarized in Table 5.3.b below (**GNI Growth Rate):**

Period	1962-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	Yes	Yes	Yes	No
R-Squared Criteria	Maybe (.5513)	No (.2638)	Yes (.6134)	NA

Table 5.3.b *India GNI Growth Rate 1961-2020*

5.6 India Manufacturing as % of GDP Data 1961-2020

Year	% of GDP
1960	14.75%
1961	15.35%
1962	15.86%
1963	15.75%
1964	14.85%
1965	15.01%
1966	14.50%
1967	13.23%
1968	13.52%
1969	14.15%
1970	14.46%
1971	14.98%
1972	15.10%
1973	15.02%
1974	16.35%
1975	15.84%

Year	% of GDP
1976	16.27%
1977	16.08%
1978	17.10%
1979	17.85%
1980	16.75%
1981	16.77%
1982	16.37%
1983	16.66%
1984	16.71%
1985	16.42%
1986	16.22%
1987	16.21%
1988	16.10%
1989	16.90%
1990	16.60%

Year	% of GDP
1991	15.68%
1992	15.80%
1993	15.92%
1994	16.76%
1995	17.87%
1996	17.60%
1997	16.52%
1998	15.72%
1999	15.18%
2000	15.93%
2001	15.31%
2002	15.56%
2003	15.59%
2004	15.83%
2005	15.97%

Year	% of GDP
2006	17.30%
2007	16.86%
2008	17.10%
2009	17.14%
2010	17.03%
2011	16.14%
2012	15.82%
2013	15.25%
2014	15.07%
2015	15.58%
2016	15.16%
2017	15.02%
2018	14.85%
2019	13.33%
2020	12.96%

Table 5.4.a *India Manufacturing as % of GDP 1961-2020*

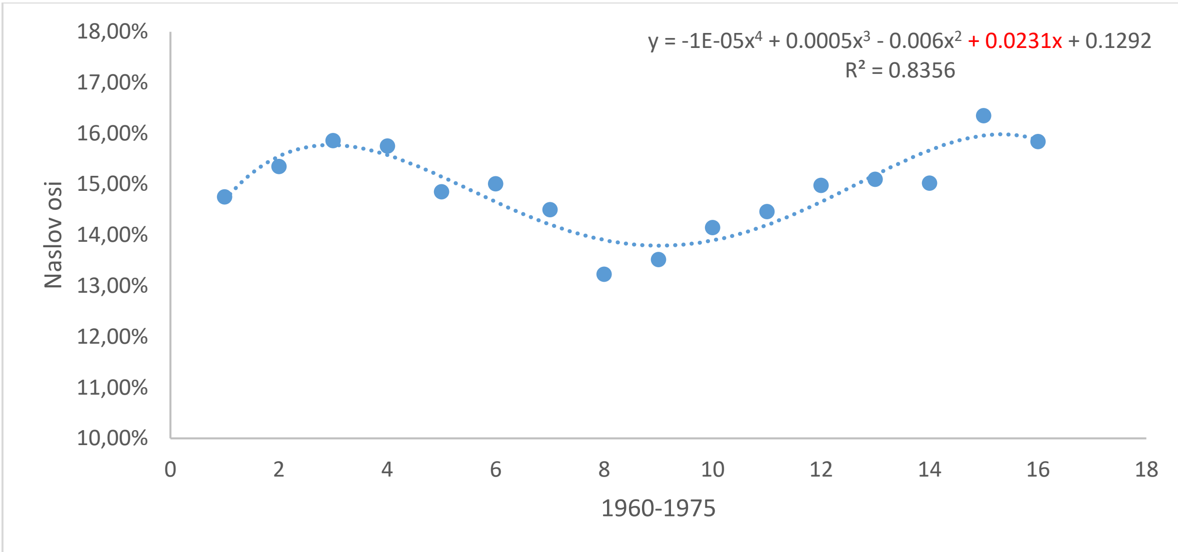


Figure 5.4.a *India Manufacturing as % of GDP 1960-1975*

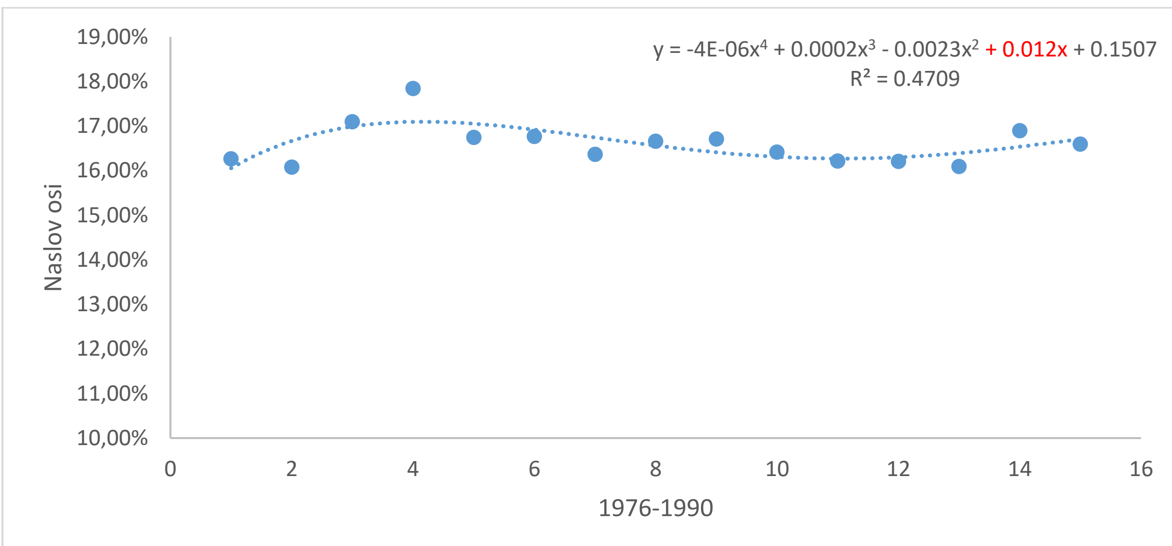


Figure 5.4.b *India Manufacturing as % of GDP 1976-1990*

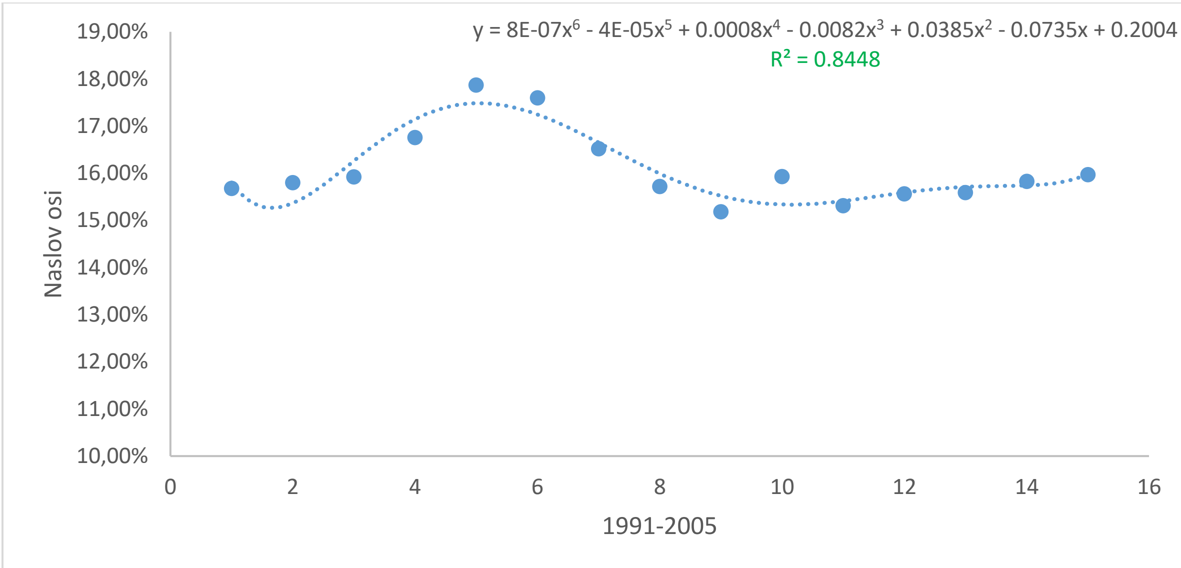


Figure 5.4.c *India Manufacturing as % of GDP 1991-2005*

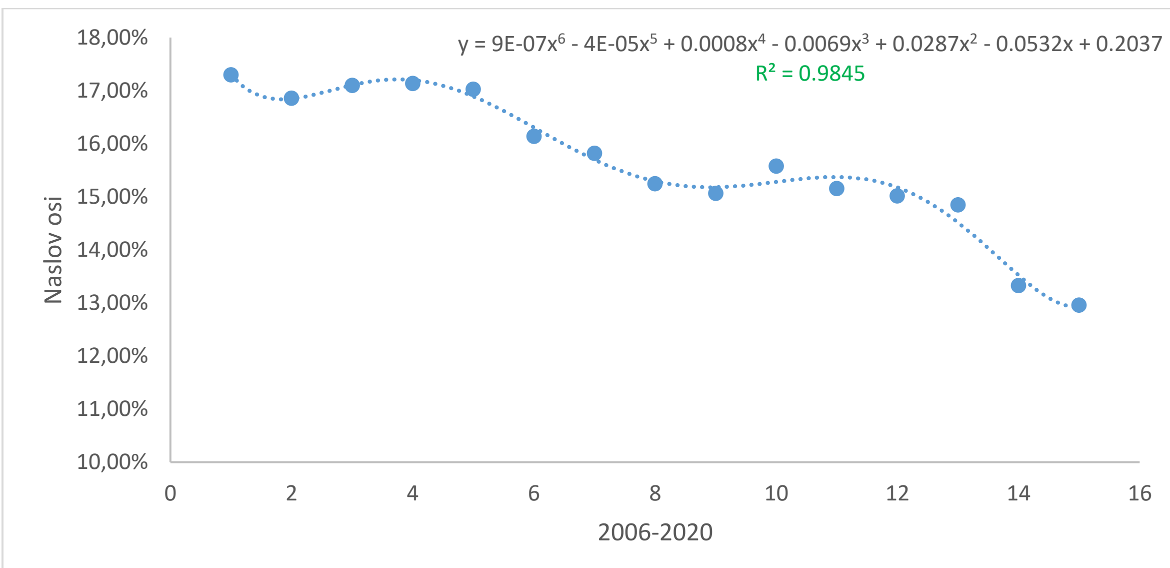


Figure 5.4.d *India Manufacturing as % of GDP 2006-2020*

The results of the above nonlinear curve fittings, based on the criteria explained in section 5.1, are summarized in Table 5.4.b below (**Manufacturing as % of GDP**):

Period	1960-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	No	No	Yes	Yes
R-Squared Criteria	NA	NA	Yes (.8448)	Yes (.9845)

Table 5.4.b *India Manufacturing as % of GDP 1960-2020*

5.7 India Economic Collated Results for J-Curve Analysis 1961-2020

The above results can together be collated and observed for J-Curve behavior and validation. The criteria for selection in J-Curve classification are: a) the polynomial curve fitting should satisfy the property of the Laguerre polynomials (of having coefficients of opposite signs) and b) the R-squared value should be more than 0.6.

Trade Balance Growth Rate

Period	1961-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	No	Yes	Yes	No
R-Squared Criteria	NA	Yes (.8885)	Yes (.8797)	NA

GDP Growth Rate

Period	1961-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	Yes	Yes	Yes	No
R-Squared Criteria	No (.3975)	No (.2809)	Yes (.6026)	NA

GNI Growth Rate

Period	1962-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	Yes	Yes	Yes	No
R-Squared Criteria	Maybe (.5513)	No (.2638)	Yes (.6134)	NA

Manufacturing as % of GDP

Period	1960-1975	1976-1990	1991-2005	2006-2020
Polynomial Criteria	No	No	Yes	Yes
R-Squared Criteria	NA	NA	Yes (.8448)	Yes (.9845)

Table 5.5 *India Economic Liberalization J-Curve Validation*

5.8 Key Results

1. It is clear from the above analysis that, as was proposed in the beginning of this chapter, liberalization of the Indian economy did in fact manifest the J-Curve behavior for the a) Trade Balance Growth Rate, b) GDP Growth Rate, c) GNI Growth Rate and d) Manufacturing as % of GDP.
2. The J-Curve is observed following the opening up of the economy in 1991 (for the 15 years window that was studied). This J-Curve behavior is not observed in the years preceding 1991, nor in the years following 2005 – <u>for all the four economic parameters considered</u>.
3. All the four economic metrics studied satisfied the J-Curve conditions only for the period 1991-2005, as was proposed. This mathematically vindicates the description of J-Curve behavior mentioned in the context of India's post liberalization (Hashim *et al.*, 2011; Virmani and Hashim, 2011).
4. The GNI growth rate showed a possible J-Curve behavior for the period 1962-1975 (the other three parameters did not) since its R-squared value was reasonably close to the cutoff. It is open-ended as of now if GNI growth rate did in fact follow the J-Curve behavior and if yes, what was the cause.
5. The Trade Balance showed a strong J-Curve behavior in the period 1976-1990 (the other three parameters did not) – it would be interesting to study further if there was an underlying cause, which led to a possible J-Curve behavior.
6. The Manufacturing as a % of GDP showed a strong J-Curve behavior in the period 2006-2020 (the other three parameters did not) – it would be interesting to study further if there was an underlying Cause which led to a possible J-Curve behavior.

CHAPTER VI:

CASE STUDY 2: J-CURVE IN RETURNS FOR VENTURE INVESTMENTS

6.1 J-Curve in Returns on Venture Funding

The Cumulative-Cash-Flow diagram for a new venture, as a function of time is observed to follow the J-Curve (Hamermesh et al., 2002; Fig. 1). Meyer and Mathonet (2011) studied the Internal Rate of Return (IRR) of Private Equity (PE) funds' investments over a 10-year window and showed that IRR follows the J-Curve. Kim and Kim (2020) observed that revenues in a startup cycle follow a J-Curve.

The IRR has been well explained, including some of the limitations and considerations, in the context of venture capital investments (Kellner, 2018), including a pictorial representation of net cash as a function of cash inflows and outflows (see Fig. 6.1 below):

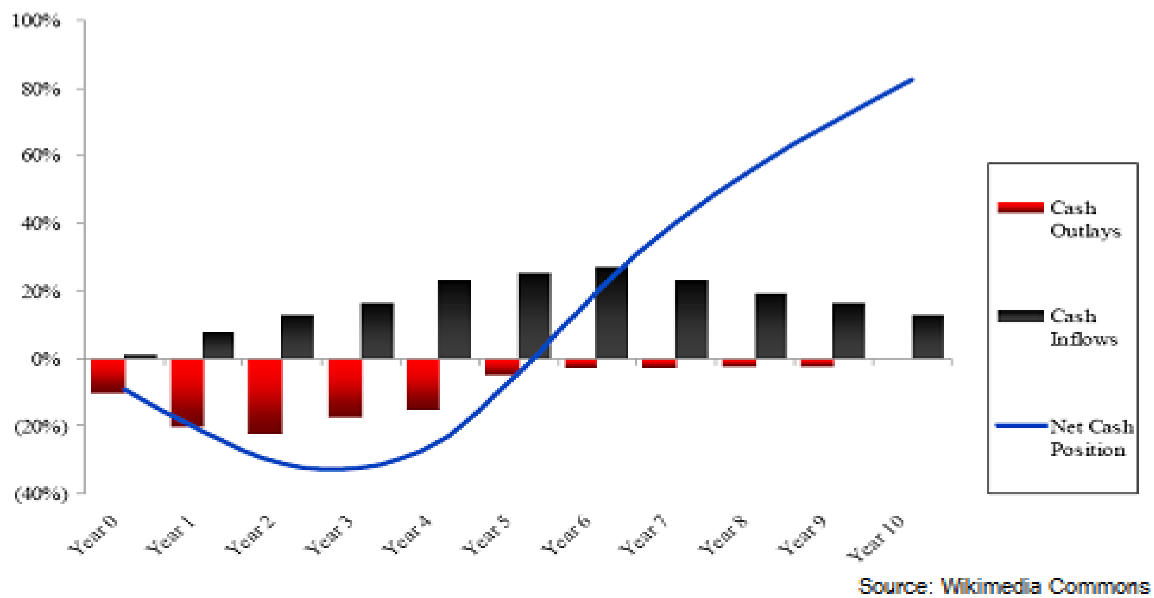


Figure 6.1 *Net Cash in Venture Investments*

Based on the mathematical modelling of the J-Curve presented in this thesis, it will be interesting to see if the above observations and conclusions can be mathematically and quantitatively substantiated (as per process defined in Figure 4.3).

6.2 Data for Returns on Venture Funding

(Swildens and Yee, 2019) studied the cash inflows on venture funding model over a 12 year period for investment in 20 companies including management fees and interest. The data is given below from their study.

Year	0	1	2	3	4	5	6	7	8	9	10	11	12
Net Returns	0	-21.4	-21.5	-14.4	-12.7	-3.3	7.6	20.9	28.5	32.2	56.1	69	69

Table 6.1 *Net Returns of Venture Investments*

6.3 Analysis for J-Curve for Venture Investments

The curve fitting of the above data is given in Fig. 6.2 below:

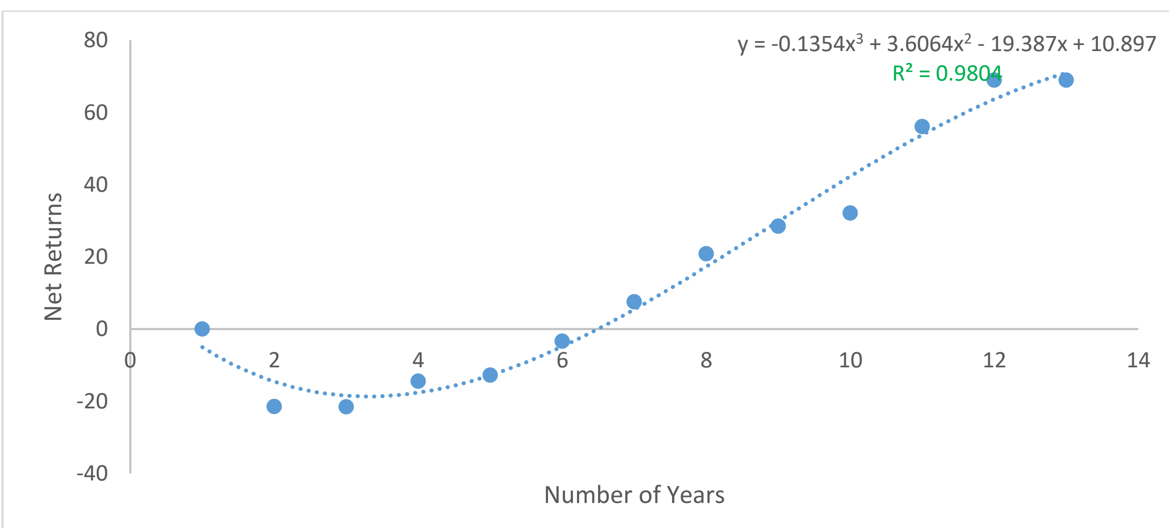


Figure 6.2 *Net Returns on Venture Investments*

6.4 Key Results

It is observed that the net returns on venture investments over a 10-12 year period do indeed satisfy the mathematical criteria of J-Curve, as expected.

We have given a satisfactory mathematical vindication of the phenomenon, which visually followed the J-Curve.

CHAPTER VII:

CASE STUDY 3: J-CURVE IN RETURNS ON STOCK INVESTMENTS

7.1 J-Curve in Returns on Stock Investments

It is that from a long term investment strategy perspective, it is advisable to hold investments for more than one year to get higher rewards on investments in financial instruments like stock markets, mutual funds, bonds, etc. (Dearking, 2021).

Also, returns on investments are expected to follow the J-Curve behavior ("Investment | J-Curve Effect", n.d.), see Fig. 7.1.

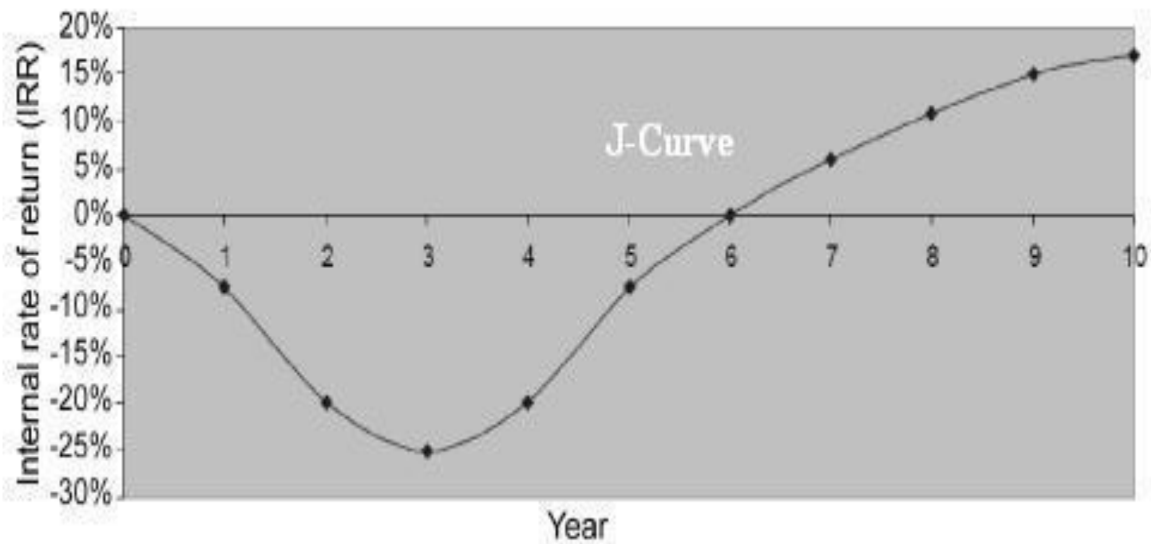


Figure 7.1 *J-Curve and Return on Investments ("Investment | J-Curve Effect", n.d.)*

It will be interesting to see if this can be substantiated and validated by looking at stock market over a period, and if there is any connection to the J-Curve phenomenon, (this is likely since it is speculated that the initial returns may not be good but will pick up over time – a typical J-Curve characteristic).

Tracy (2020) describes the J-Curve in the context of equity investments.

“The *J-curve effect* is a phenomenon in which a period of negative or unfavorable returns is followed by a gradual recovery that stabilizes at a higher level than before the decline. The J-curve effect is often seen in a country's balance of trade and equity fund returns.”

Kenton (2021) explains:

“J Curves demonstrate how private equity funds historically usher in negative returns in their initial post-launch years but then start witnessing gains after they find their footing. Private equity funds may take early losses because investment costs and management fees initially absorb money. But as funds mature, they begin to manifest previously unrealized gains, through events such as mergers and acquisitions (M&A), initial public offerings (IPOs), and leveraged recapitalization.”

Towards this, data from the Indian stock market will be studied for the period 2016-2021 for the NIFTY500 index listed on India’s National Stock Exchange (NSE). The reasons for choosing this index are as follows (“NSE - National Stock Exchange of India Ltd.”, n.d.).

“It represents the top 500 companies based on full market capitalization from the eligible universe. The NIFTY 500 Index represents about 96.1% of the free float market capitalization of the stocks listed on NSE as on March 29, 2019. The total traded value for the last six months ending March 2019, of all Index constituents is approximately 96.5% of the traded value of all stocks on NSE.”

7.2 Data for Returns on Stock Investments

The data for the NIFTY 500 index have been obtained from the historical records of India’s National Stock Exchange (“NSE - National Stock Exchange of India Ltd.”, n.d.).

The following considerations have been kept in mind to accumulate and analyze the data:

a) Calculate returns on investments on the stock market returns for the NIFTY 500 index, from the supposed date of investment (taken as 01Jan). Calculate each days return % as

$$\% \, returns = \left\{ \frac{[(yesterday's\, returns - today's\, returns)]}{yesterday's\, returns} \right\} * 100 \quad 7.1$$

b) Calculate the average percentage (%) returns for the week/month etc. from the daily % returns as per the above formula eqn. 7.1.

c) Study data for a two-year window, since it important to study returns over a period more than one year.

d) Choose four different windows of two years periods to prevent any time/period specific features from distorting the analysis,:

    1) Jan2016 – Dec2017,
    2) Jan2017 – Dec2018,
    3) Jan2018 – Dec2019,
    4) Jan2019 – Dec2020.

e) Assume the investment date to be 01Jan for each of the four windows.

f) For each of the windows, calculate return on investments for the following periods, from the assumed date of investment (taken as 01Jan). As an example, for the 2016-2017 window, the periods studied for returns on investment will be:

    1) One month: January16 (1-month period)
    2) Six months: January16 – June16 (6 months period)

3) One year: January16 – December16 (12 months)

4) Two years: January16 – December17 (24 months)

Take similar data for the other three windows of 2017-2019, 2018-2020 and 2019-2021.

g) Study curves for all the four 2-year windows for the four specified periods of 16/12/24 months.

h) We expect the following behavior for the two-year % returns

   1) No J-Curve behavior should be observed for 1-month and 6-months window since the stock market is not expected to give a positive yield for less than one year.

   2) J-Curve may probably be observed for the 1-year returns, since positive yields are expected for investments post one year.

   3) J-Curve is expected for a 2-year investment period. If a J-Curve is observed, it has to be validated as per the criteria for J-Curve behavior outlined in Figure 4.3.

i) If the J-Curve is mathematically validated for returns beyond 1-year period, it can be postulated that J-Curve is observed in stock market returns for investments beyond 1-year period.

j) One year of the pandemic 2020-2021 have also been included in this study – to see if the stock markets behaved significantly differently from the earlier years.

k) It is expected that a J-Curve behavior will manifest for investment periods greater than one year.

7.3 Relevant Data of NIFTY 500 % Returns for 2016-2021

Percentage (%) returns for January 2016, 2017, 2018, 2019 (1 month after investment on 01 January of each year) are given in Table 7.1:

% Returns Jan2016	
60.2565	-3.1882
3.10222	3.22171
6.681108	-15.525
0.878966	14.9951
-12.9572	4.60926
2.553901	-15.197
-2.46125	-18.323
38.1273	17.9593
-13.5017	40.6032
	1.14863

% Returns Jan2017	
14.32128	12.079729
12.89991	12.683274
9.272401	-6.09027
-4.02466	10.907243
-21.2876	-9.907787
22.99761	10.367988
22.18847	59.895573
-15.4163	-20.08734
1.557969	-19.08892
-18.9521	36.763624

% Returns Jan2018	
13.97403	15.50993
3.114029	3.35834
8.257567	13.0295
9.341335	-22.2451
-4.77638	8.707808
-2.92147	19.28562
0.833578	0.571905
-3.17283	10.06243
8.19454	-24.7572
-9.86653	-5.88333
	12.20205

% Returns Jan2019	
59.37954	4.885258
-4.55328	4.745622
3.466708	2.226657
-12.7145	6.944588
6.052605	-12.2219
14.07677	2.542777
-19.8334	24.51813
2.927503	0.199416
1.949109	-4.71296
12.03704	4.489439
-6.00075	27.60081

Table 7.1 *Percentage (%) returns for January (1 month) starting 2016, 2017, 2018, 2019*

Average % returns for January-June 2016, 2017, 2018, 2019 (6 months after investment on 01 January of each year) are given in Table 7.2:

% Returns Jan16-Jun16	
8.2934667	2.756753
5.5990091	3.028982
5.9465494	2.592645
4.0442419	2.900929
4.4465872	2.45009
2.2244445	2.765417
4.6709829	2.358162
3.529155	5.946549
2.8050844	3.163708
2.9527394	2.734601
2.9428789	2.694509
	3.381577

% Returns Jan17-Jun17	
5.118896	2.775896
3.795504	2.735432
5.55401	2.756405
3.83209	2.752288
2.648705	2.400098
3.470682	2.312322
4.086492	2.213302
3.192404	2.880927
3.385919	2.342916
3.114672	2.301186
3.194235	2.124192
	2.159989

% Returns Jan18-Jun18	
3.081232	1.8361579
2.330756	1.6645079
2.515234	1.7837518
1.891528	1.6629524
1.417094	1.6389082
1.911475	1.5314213
1.76562	1.5356379
1.266027	2.5388946
2.581706	1.8257434
1.80151	1.7048367
2.559327	1.8068221
	1.8932905

% Returns Jan19-Jun19	
6.100237	2.1002853
5.039294	1.9567908
5.36387	1.9552567
3.20389	2.3876801
3.175086	2.0452237
2.266481	1.9660478
3.248457	2.4879584
2.247615	2.3893036
3.238887	1.9233731
2.243628	1.9254669
2.774024	1.8381746
	2.0331858

Table 7.2 *Percentage (%) returns for January-June (6 months) starting 2016, 2017, 2018, 2019*

Average % Returns for January-December 2016, 2017, 2018, 2019 (12 months after investment on 01 January of each year) are given in Table 7.3:

% Returns Jan16-Dec16	% Returns Jan17-Dec17	% Returns Jan18-Dec18	% Returns Jan19-Dec19
5.9465494	5.554009763	2.51523378	5.36387011
4.67098291	4.086491995	1.765619977	3.24845673
2.94287893	3.194234783	2.559326637	2.7740236
2.90092947	2.752287603	1.662952428	2.3876801
5.9465494	2.880926708	2.538894614	2.38930356
3.38157661	2.159989272	1.893290549	2.03318577
2.95802865	4.093667521	1.819075814	1.95837405
3.14912707	3.847625503	1.647868242	1.8096365
3.03591763	3.415790468	1.933148044	2.39165116
5.83224775	5.234651345	1.766087749	5.22478552
5.63868288	5.077762387	4.451245736	5.17973695
5.23213438	4.625486306	4.034645375	4.65516328

Table 7.3 *Percentage (%) returns for January-December (1 year) starting 2016, 2017, 2018, 2019*

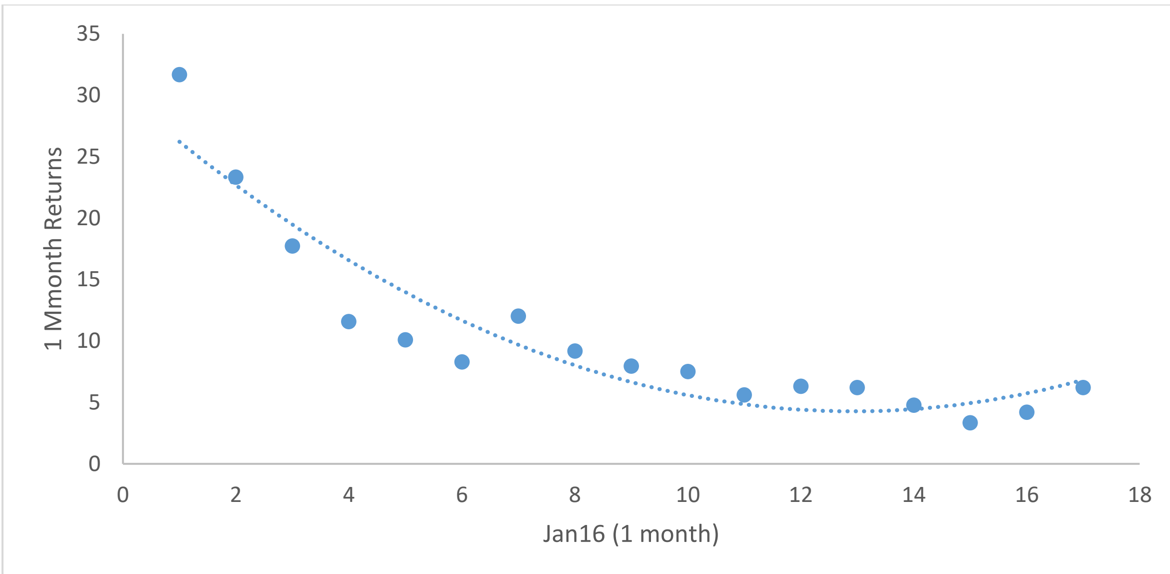
Average percentage (%) returns for January to December of next year for 2016, 2017, 2018, 2019 (24 months after investment on 01 January of preceding year) are given in Table 7.4:

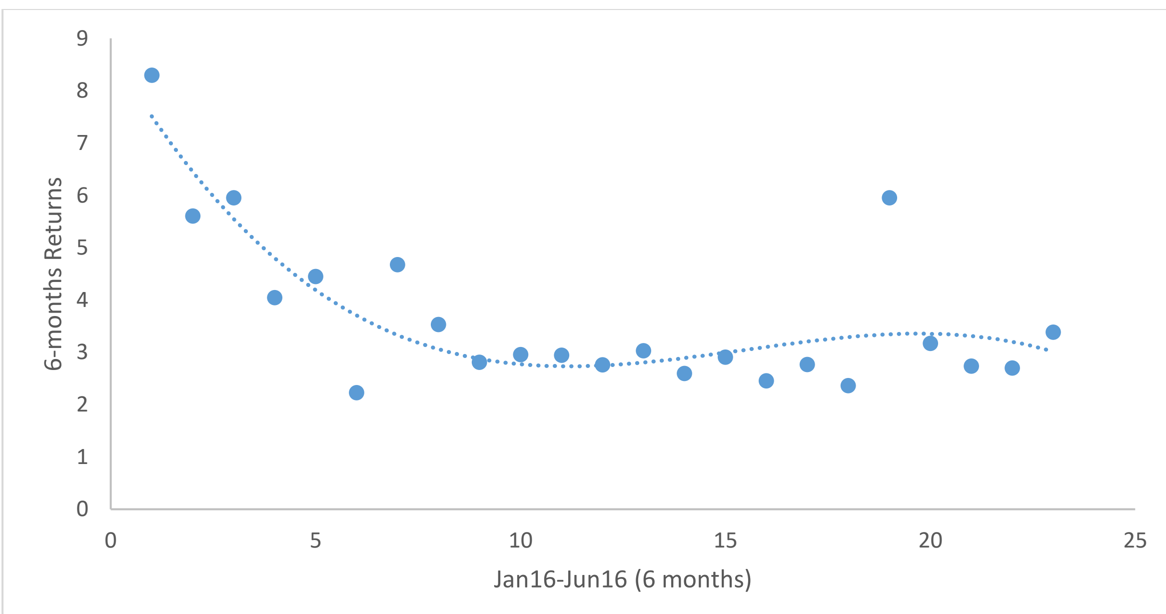
% Returns Jan16-Dec17	% Returns Jan17-Dec18	% Returns Jan18-Dec19	% Returns Jan19-Dec20
5.9465494	5.554009763	2.51523378	5.36387011
4.67098291	4.086491995	1.765619977	3.24845673
2.94287893	3.194234783	2.559326637	2.7740236
2.90092947	2.752287603	1.662952428	2.3876801
5.9465494	2.880926708	2.538894614	2.38930356
3.38157661	2.159989272	1.893290549	2.03318577
2.95802865	4.093667521	1.819075814	1.95837405
3.14912707	3.847625503	1.647868242	1.8096365
3.03591763	3.415790468	1.933148044	2.39165116
5.83224775	5.234651345	1.766087749	5.22478552
5.63868288	5.077762387	4.451245736	5.17973695
5.23213438	4.625486306	4.034645375	4.65516328
5.19785494	4.362702604	4.015201216	4.51739128
5.02139951	4.136707952	3.795009102	4.36089014
4.77790713	4.152881015	3.663926048	4.06732093
4.58336747	3.86275022	3.514033803	4.074887
4.50494446	3.96651446	3.44185143	4.04534501
4.17416196	3.677430371	3.270040856	3.73060549
4.77472187	3.549350463	3.167565091	3.6013573
4.64174612	3.441619963	3.047097326	3.55998805
4.5075258	3.438609021	3.242857213	3.31568524
5.19878451	3.293509068	4.535152151	3.19960255
5.12531463	4.487301729	4.510183131	4.61346228
4.8972514	4.281877964	4.256377079	4.26318839

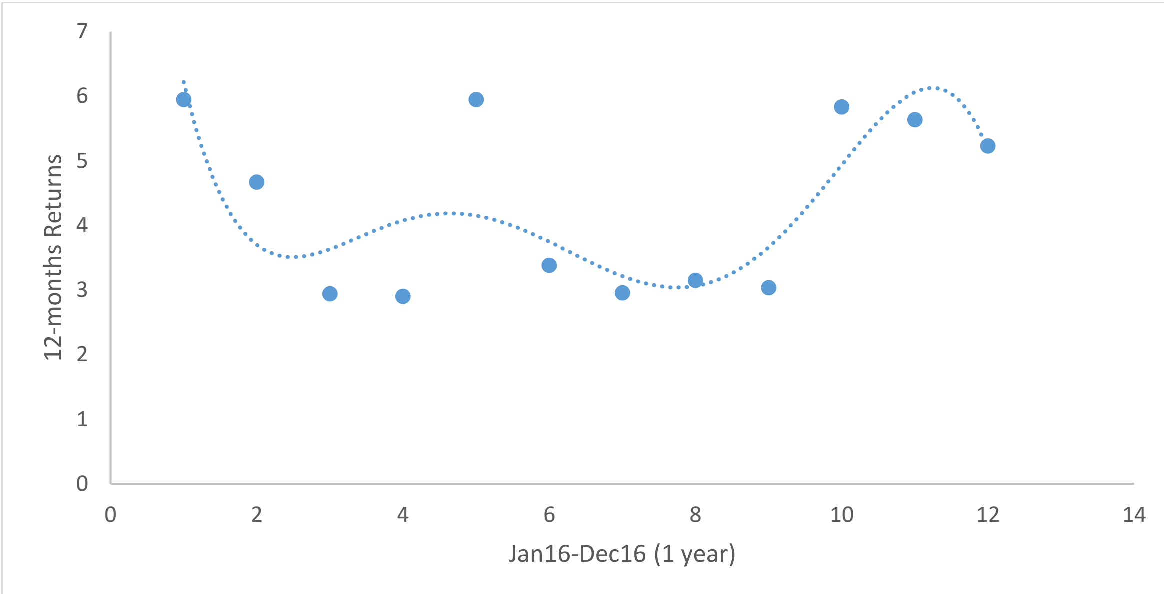
Table 7.4 *Percentage (%) returns for January-December (2 year2) starting 2016, 2017, 2018, 2019*

7.4.a Analysis of NIFTY 500 % Returns Data for 2016-2017

The percentage (%) returns for 1 month, 6 months, 1 year and 2 years from Jan16 – Dec17 for investment starting Jan16 are given below in Fig. 7.2:







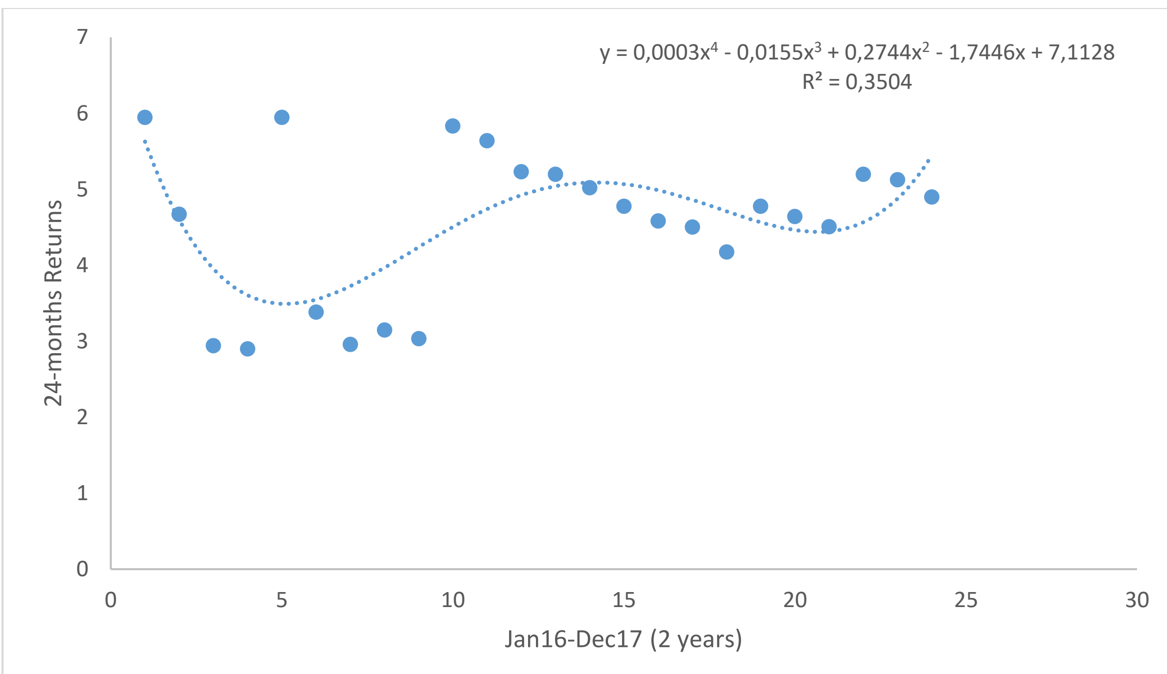
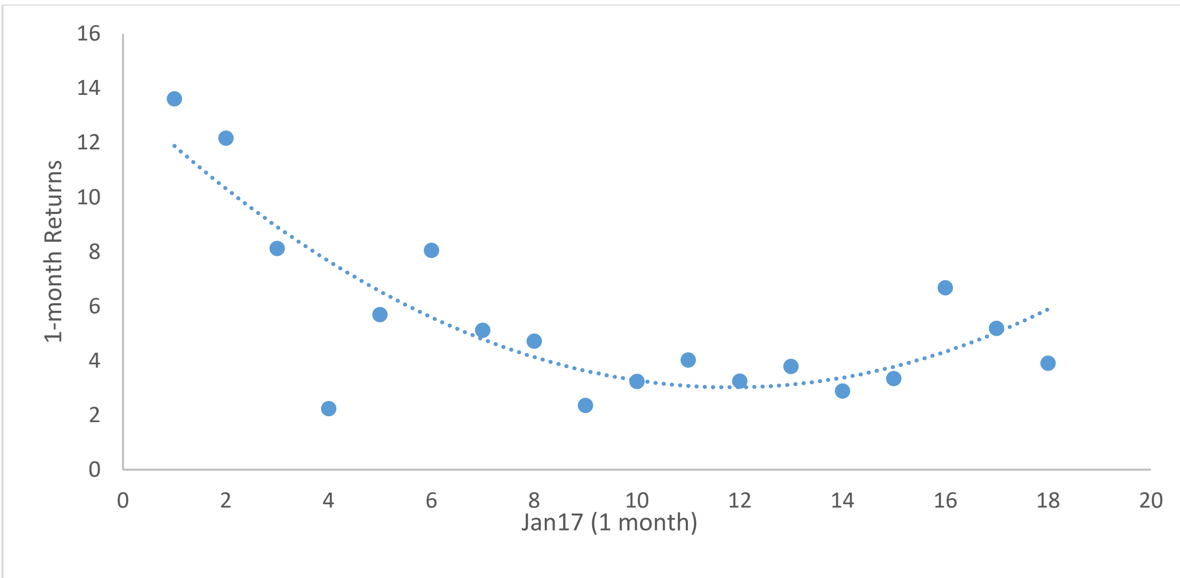
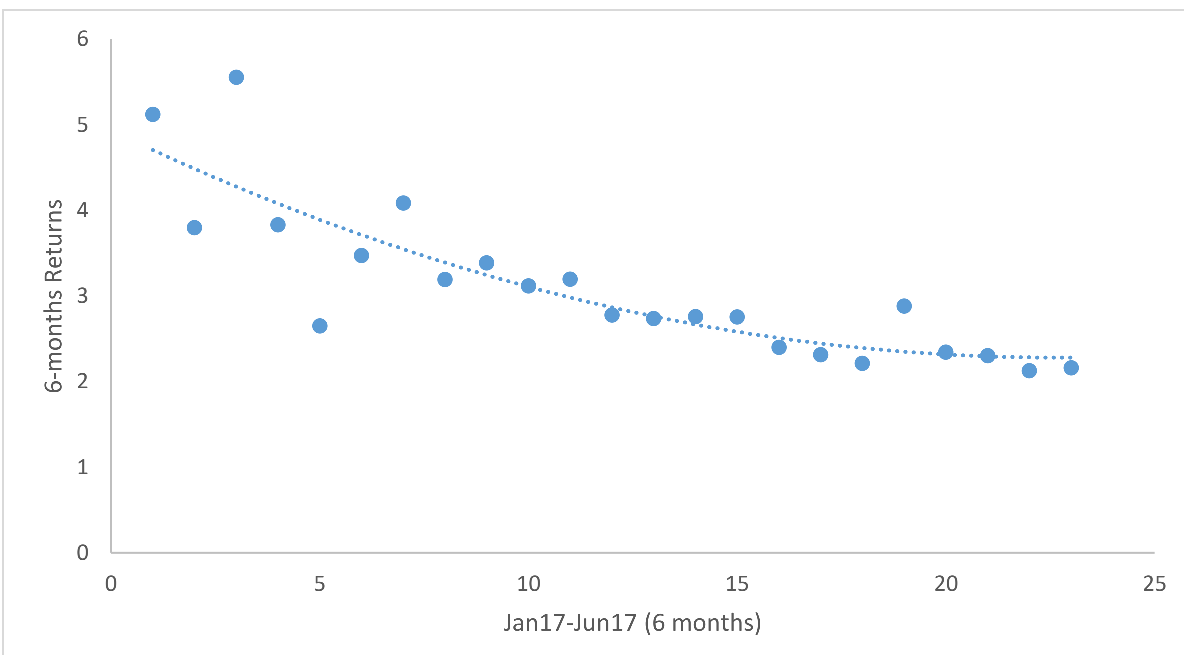


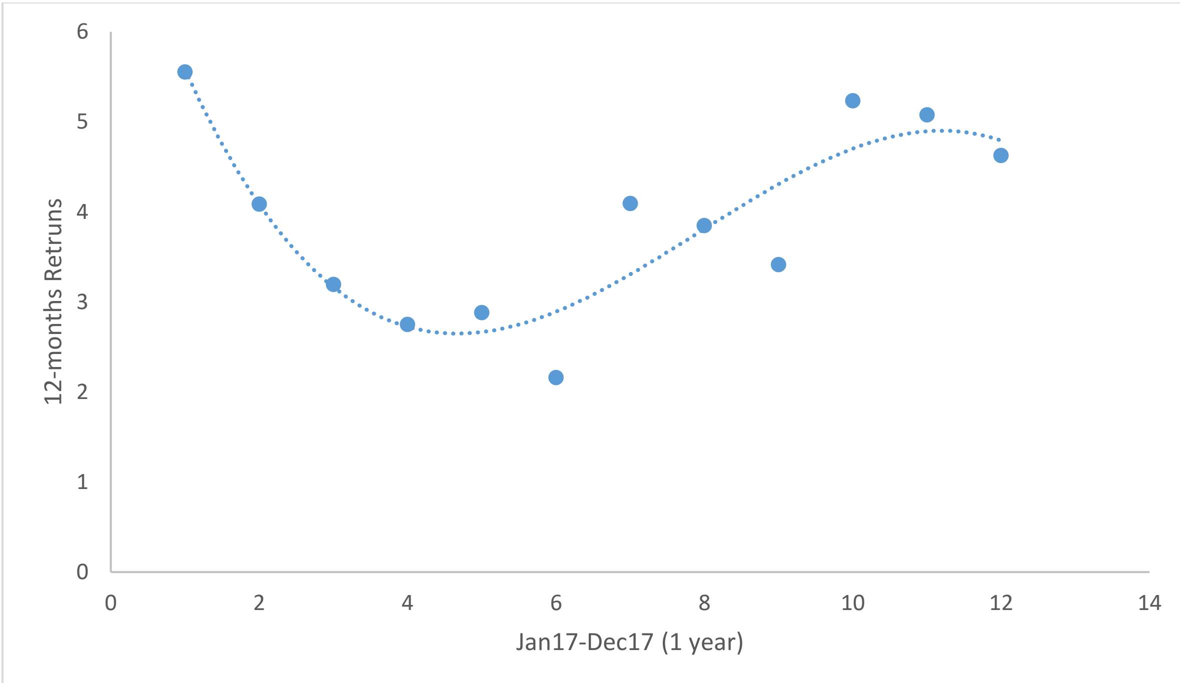
Figure 7.2 *Percentage (%) returns for 1/6/12/24 Months Starting Jan 2016*

7.4.b Analysis of NIFTY 500 % Returns Data for 2017-2018

The percentage (%) returns for 1 month, 6 months, 1 year and 2 years from Jan17 – Dec18 for investment starting Jan17 are given below in Fig. 7.3:







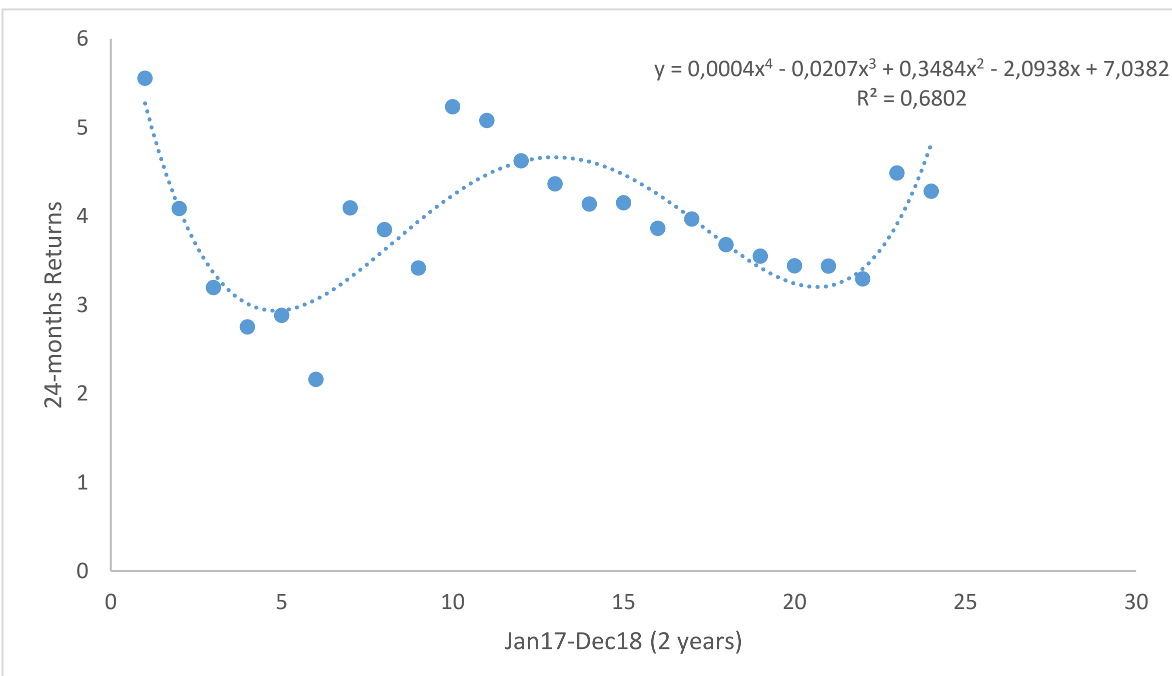
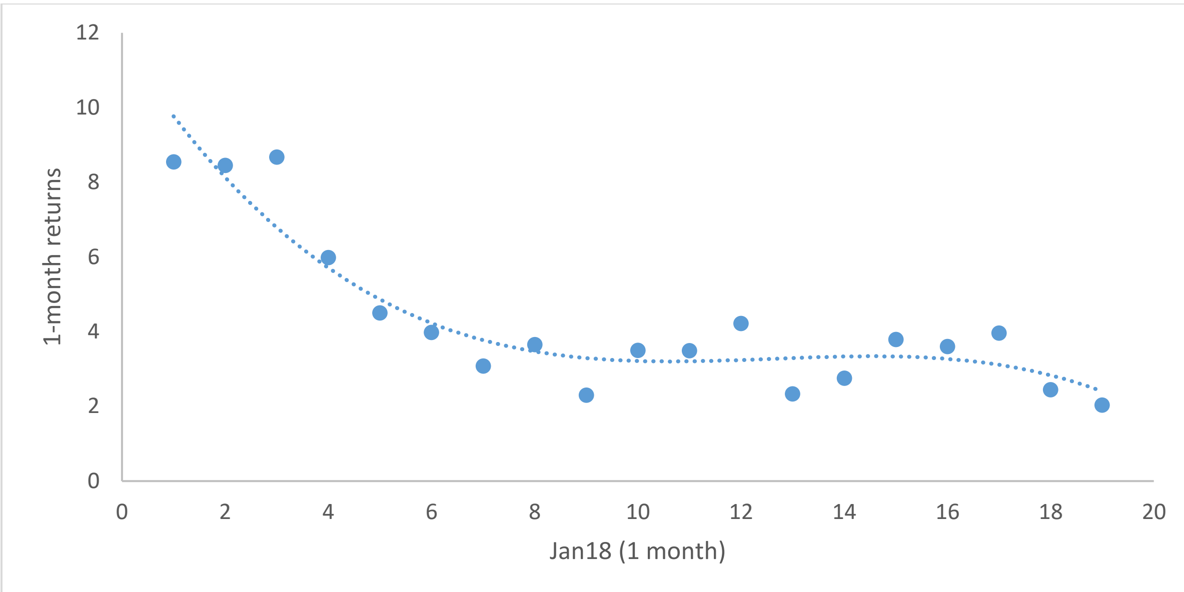
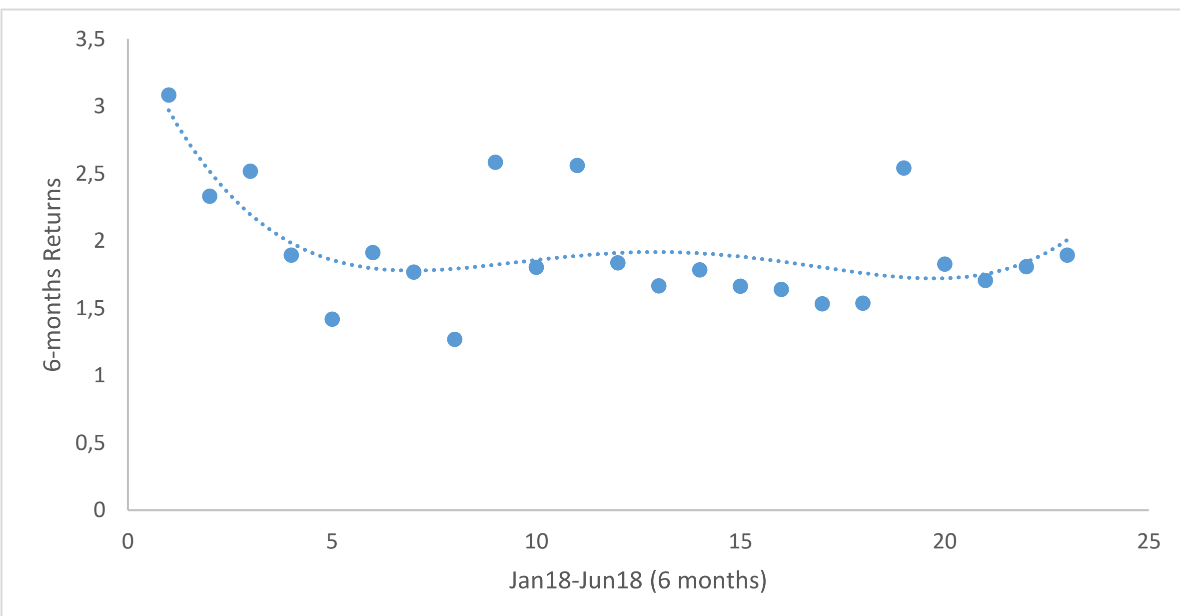


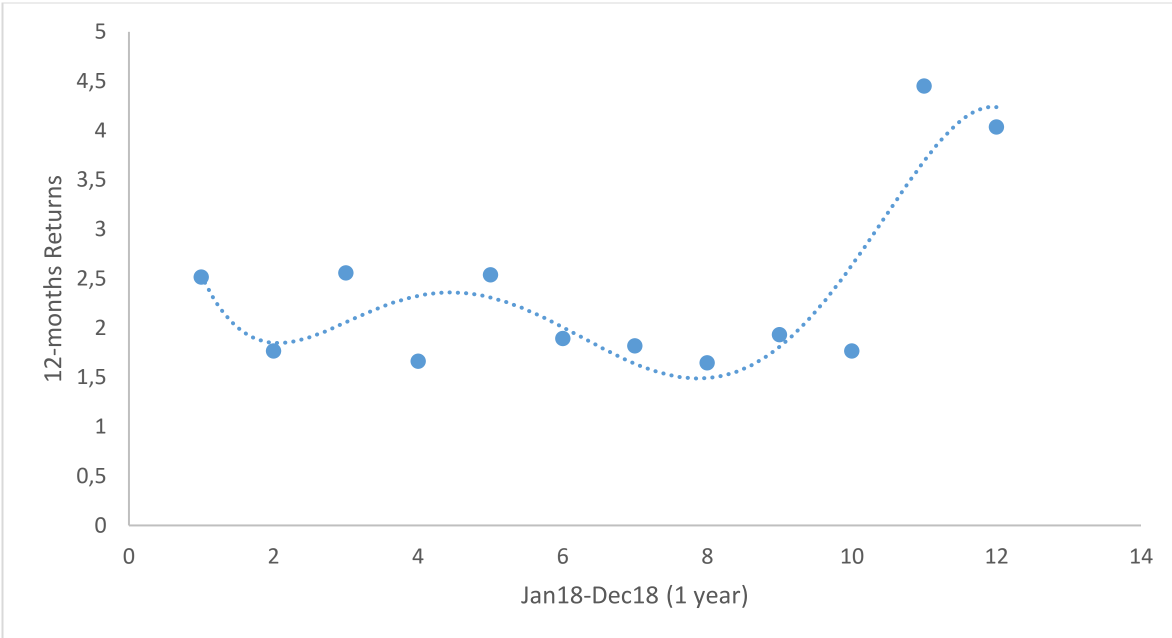
***Figure 7.3** Percentage (%) returns for 1/6/12/24 Months Starting Jan 2017*

7.4.c Analysis of NIFTY 500 % Returns Data for 2018-2019

The % returns for 1 month, 6 months, 1 year and 2 years from Jan18 – Dec19 for investment starting Jan18 are given below in Fig. 7.4:







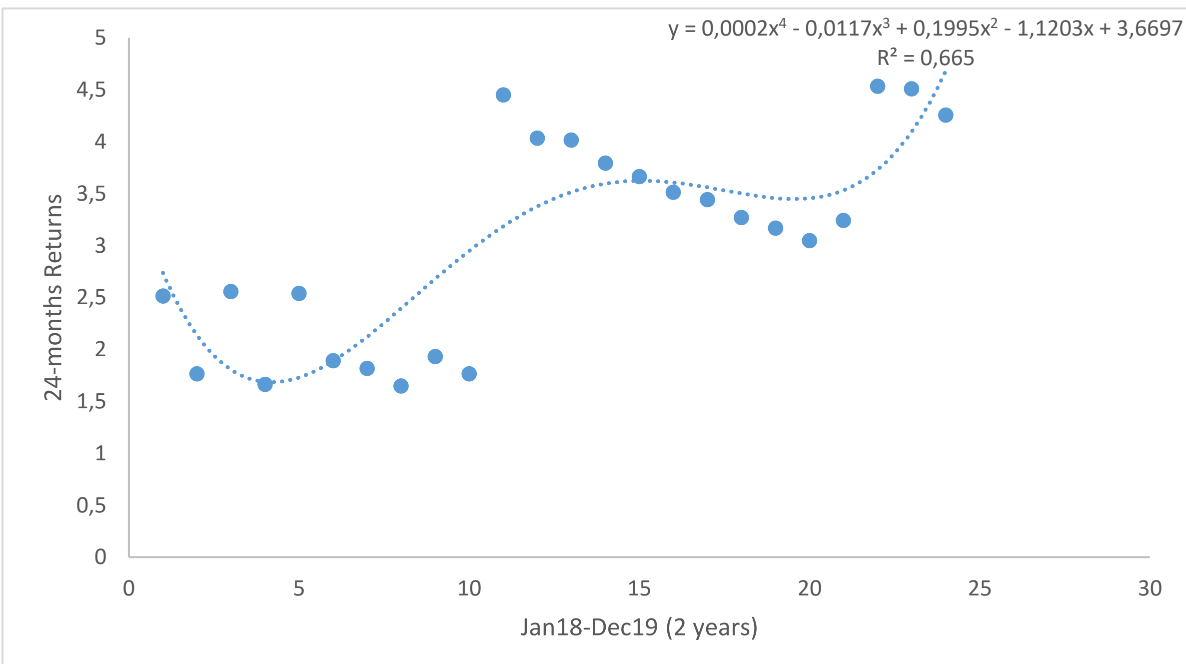
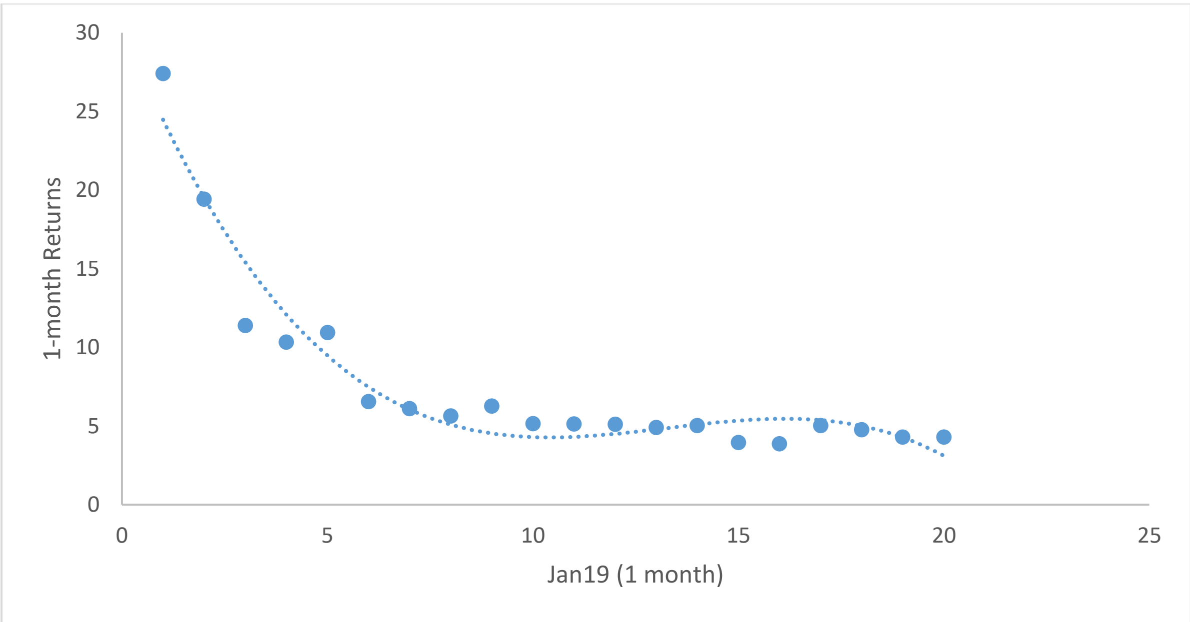
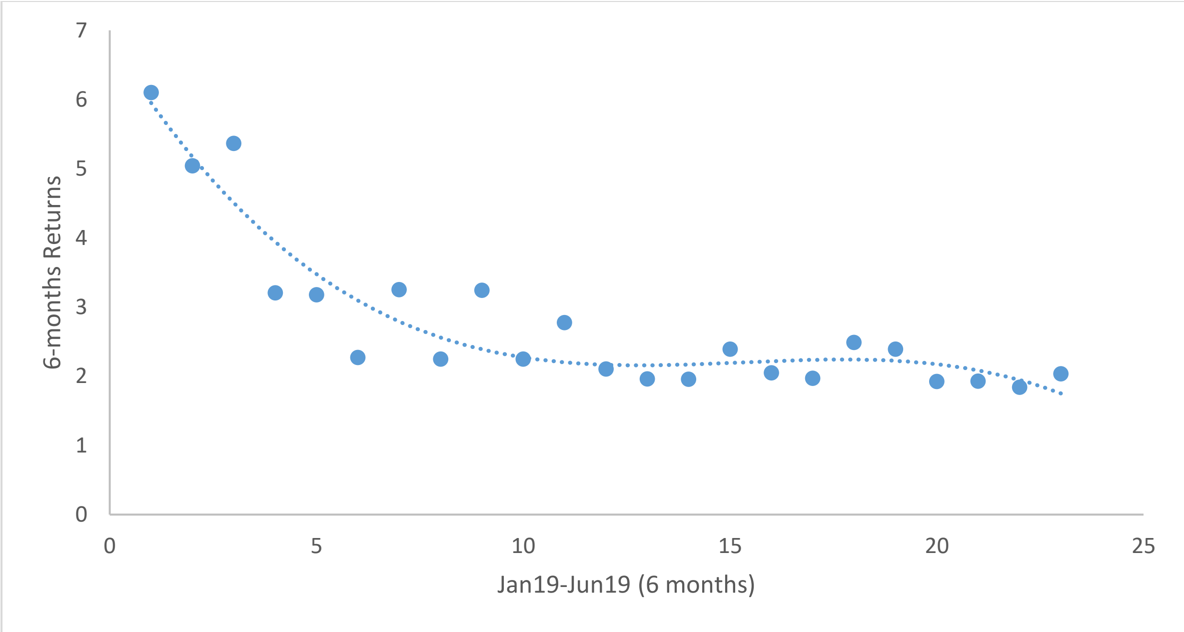


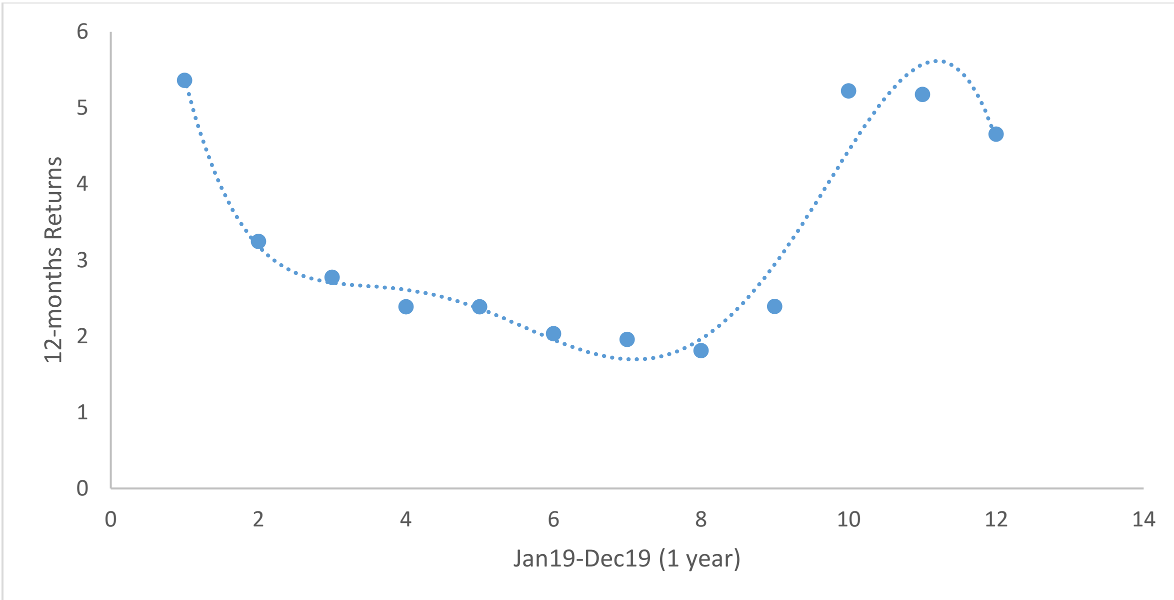
Figure 7.4 *Percentage (%) returns for 1/6/12/24 Months Starting Jan 2018*

7.4.d Analysis of NIFTY 500 % Returns Data for 2019-2020

The percentage (%) returns for 1 month, 6 months, 1 year and 2 years from Jan19 – Dec20 for investment starting Jan19 are given below in Fig. 7.5:







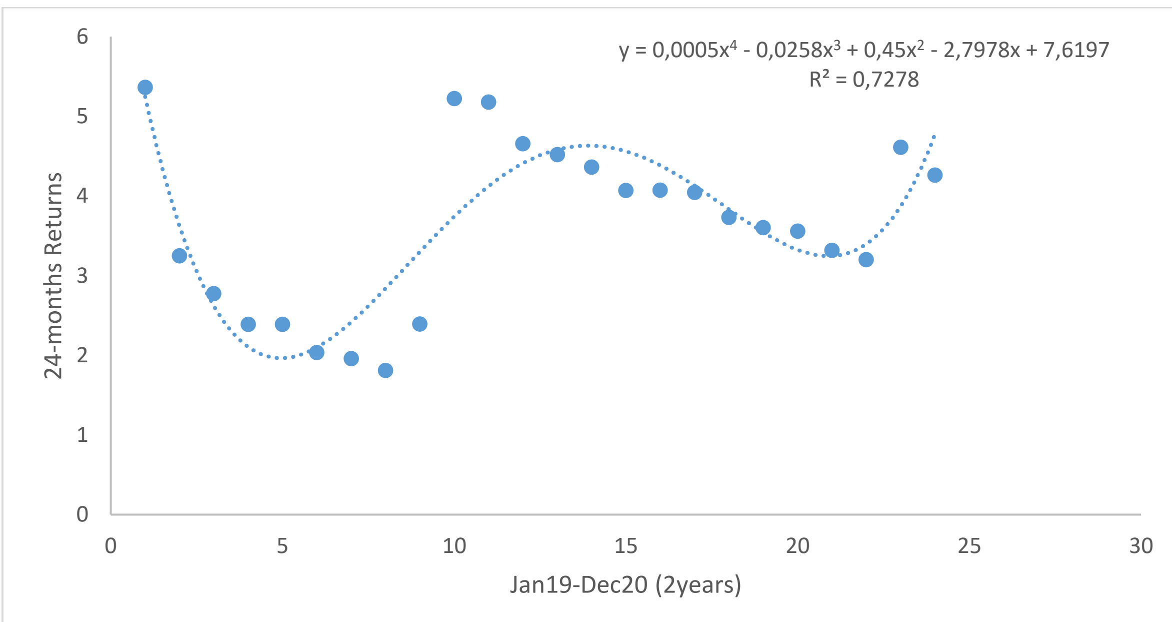


***Figure 7.5** Percentage (%) returns for 1/6/12/24 Months Starting Jan 2019*

7.5 Key Results

We studied the returns of investments in stock markets (NIFTY 500 index of India's National Stock Exchange) over a 2-year window in 4-periods of 1/6/12/24 months. To remove specific events from influencing the study four windows were chosen which overlapped with each other: 2016-17, 2017-18, 2018-19 and 2019-20.

Since it has been observed that long-term investments usually give good rewards after a one year period, it was expected to show up in the % returns over that period. It has been interesting to observe that percentage (%) returns start to show a J-Curve around the 1-year period, and do indeed show a specific J-Curve behavior at the 2-year window – and mathematically validated as per the conditions shown in Fig. 4.3.

The mathematical validation of the J-Curve for the 2-years investment period is summarized in the below table.

Period (2 years)	Jan16-Dec17	Jan17-Dec18	Jan18-Dec19	Jan19-Dec20
Polynomial Criteria	Yes	Yes	Yes	Yes
R-Squared Criteria	Maybe (.3504)	Yes (.6802)	Yes (.665)	Yes (.7278)

Table 7.5 *J-Curve Behavior of Stock Returns over 2-Years Investment Period*

It is also interesting to observe that the stock market returns during the pandemic period of Jan-Dec 2020 did not show a behavior very different from the preceding years.

CHAPTER VIII:

CASE STUDY 4: J-CURVE AND FINANCIAL CRISIS OF 2007-2008

8.1 2007-2008 Financial Crisis

Britannica gives a detailed account of the 2007-08 financial crisis (“Financial crisis of 2007–08 | Definition, Causes, Effects, & Facts | Britannica”, n.d.)

This affected some countries more than others did. It will be interesting to study the impact of this financial crisis on the GDP of the various countries and regions, and if the recovery phase displayed any J-Curve behavior as per the mathematical criteria defined in this thesis.

Towards this, we analyze the GDP data for various regions across the globe, as well as a few countries (“World GDP Data”, n.d.). We study the GDP data 2007-2019 (start of the financial crisis up to the beginning of the CoVID-19 pandemic). We chose regions across the Americas, Africa, Europe, Asia and Pacific regions, along with countries like Brail, Chile, Singapore, China and UAE.

Some clearly showed J-Curve behavior, some showed a likely probability of J-Curve, and some did not. There were also some other interesting phenomena observed.

8.2 Data for the 2007-2008 Financial Crisis

	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Australia	3.77	3.58	1.87	2.17	2.47	3.92	2.6	2.56	2.17	2.74	2.3	2.87	2.11
Brazil	6.07	5.09	-0.13	7.53	3.97	1.92	3	0.5	-3.55	-3.28	1.32	1.78	1.41
Chile	4.91	3.53	-1.56	5.84	6.11	5.32	4.05	1.77	2.3	1.71	1.18	3.71	0.94
China	14.2	9.65	9.4	10.6	9.55	7.86	7.77	7.43	7.04	6.85	6.95	6.75	5.95
UAE	3.18	3.19	-5.24	1.6	6.93	4.48	5.05	4.41	5.06	2.98	2.37	1.19	3.41
West/Central Africa	5.53	6.28	6.27	6.96	4.85	5.14	6.1	5.93	2.75	0.13	2.32	2.95	3.19
Central-Europe Baltics	6.42	3.93	-3.5	1.7	3.06	0.76	1.24	2.98	3.95	3.07	4.88	4.49	4.07
East Asia Pacific	7.47	4.39	2.56	7.61	5.47	5.18	5.24	4.75	4.71	4.66	5.05	4.73	4.06
Europe Central-Asia	3.6	1.01	-4.39	2.66	2.36	0.35	0.9	1.88	2.12	1.97	2.82	2.17	1.81
Euro Area	3	0.42	-4.52	2.18	1.72	-0.84	-0.22	1.4	2.06	1.86	2.61	1.83	1.57
Latin-America Caribbean	5.27	3.72	-2.01	6.43	4.45	2.51	2.88	1.37	0.54	-0.15	1.88	1.63	0.8
Middle-East North-Africa	5.2	4.75	0.8	5.05	3.94	3.45	2.73	2.95	2.38	4.58	1.66	1.48	1.18
North America	2.25	-0.05	-2.57	2.6	1.67	2.21	1.88	2.55	2.88	1.65	2.39	2.95	2.14
OECD	2.68	0.29	-3.33	2.92	1.89	1.37	1.53	2.17	2.59	1.85	2.51	2.36	1.74
Pacific Islands	0.99	1.69	-0.95	3.13	3.84	0.76	3.3	3.64	3.67	3.73	4.35	2.92	0.91
Singapore	9.02	1.87	0.12	14.5	6.34	4.46	4.84	3.94	2.99	3.33	4.52	3.5	1.35
Sub-Sahara Africa	6.29	5.29	3.18	5.93	4.38	3.36	5.11	4.86	2.83	1.15	2.44	2.68	2.57

Table 8.1 *Global GDP Data for 2007-2019*

8.3 Plots of Global GDP 2007-2019

We present the 2007-1019 GDP plots for seventeen countries/regions/organizations below from Fig. 8.1 to Fig. 8.17. The collated plot of all the GDPs is given in Fig. 8.18.

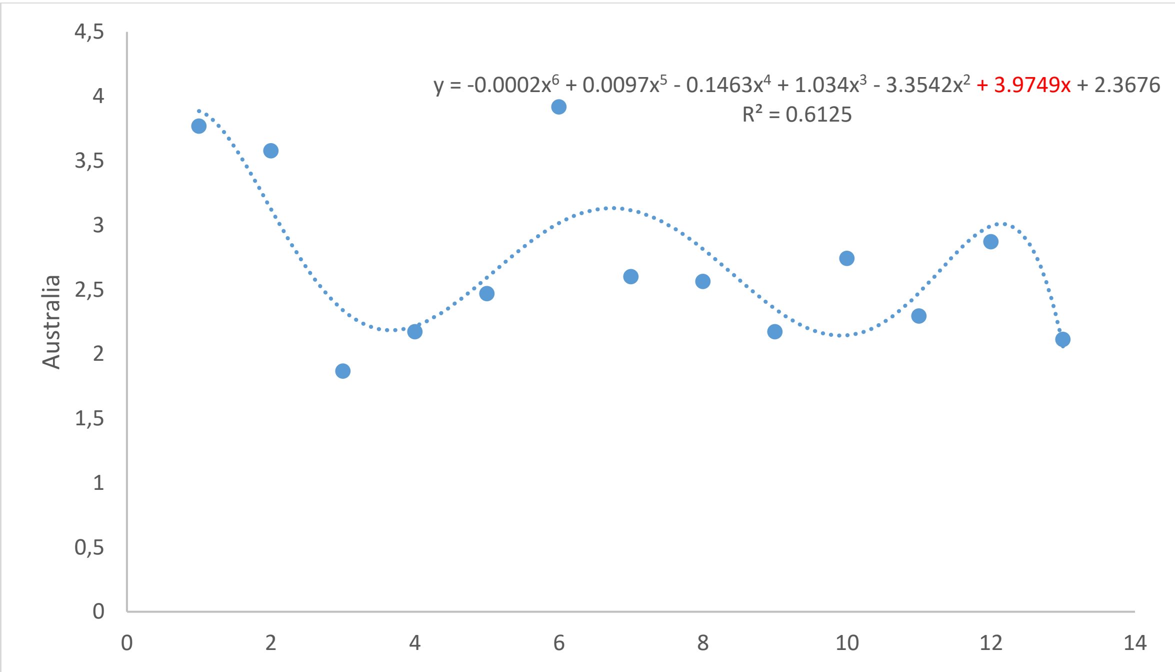


Figure 8.1 *GDP of Australia for 2007-2019*

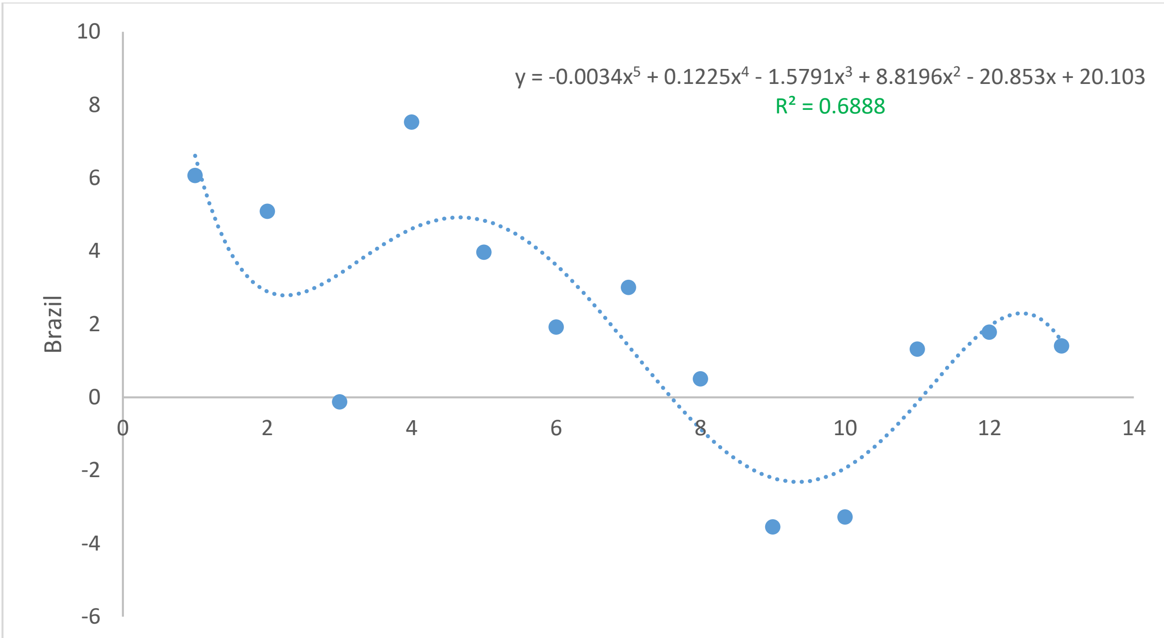


Figure 8.2 *GDP of Brazil for 2007-2019*

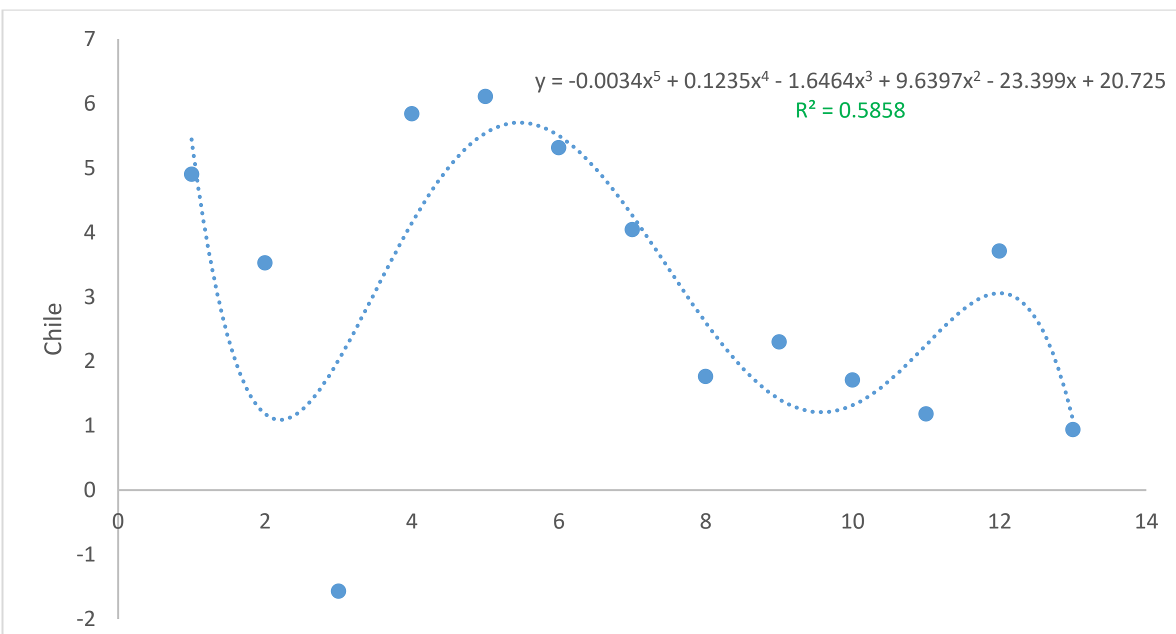


Figure 8.3 *GDP of Chile for 2007-2019*

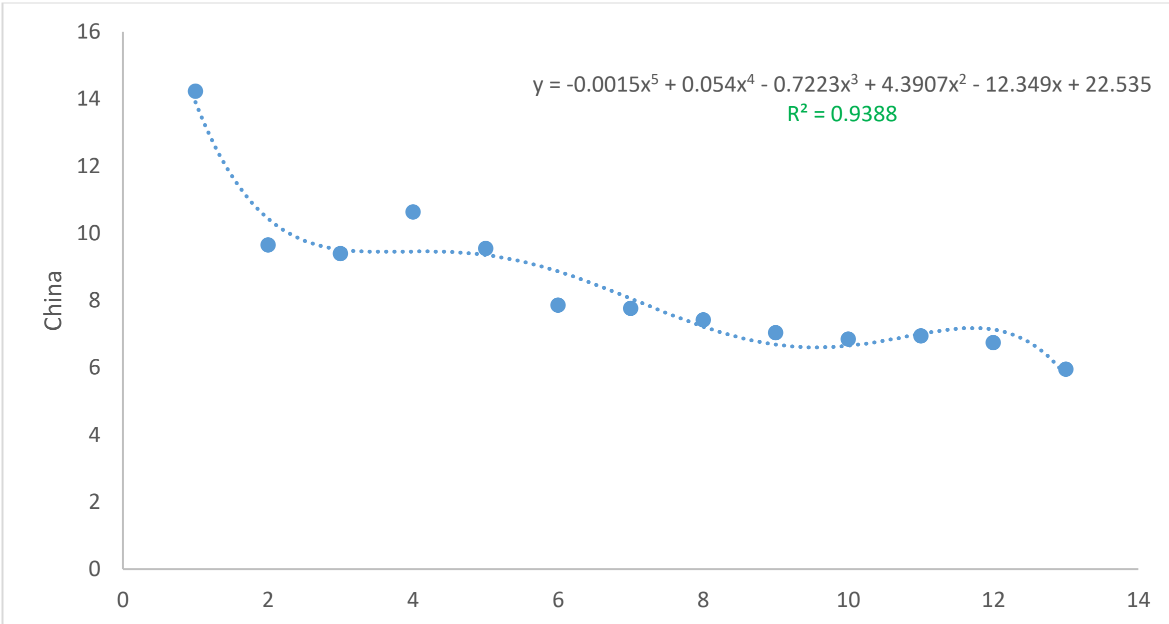


Figure 8.4 *GDP of China for 2007-2019*

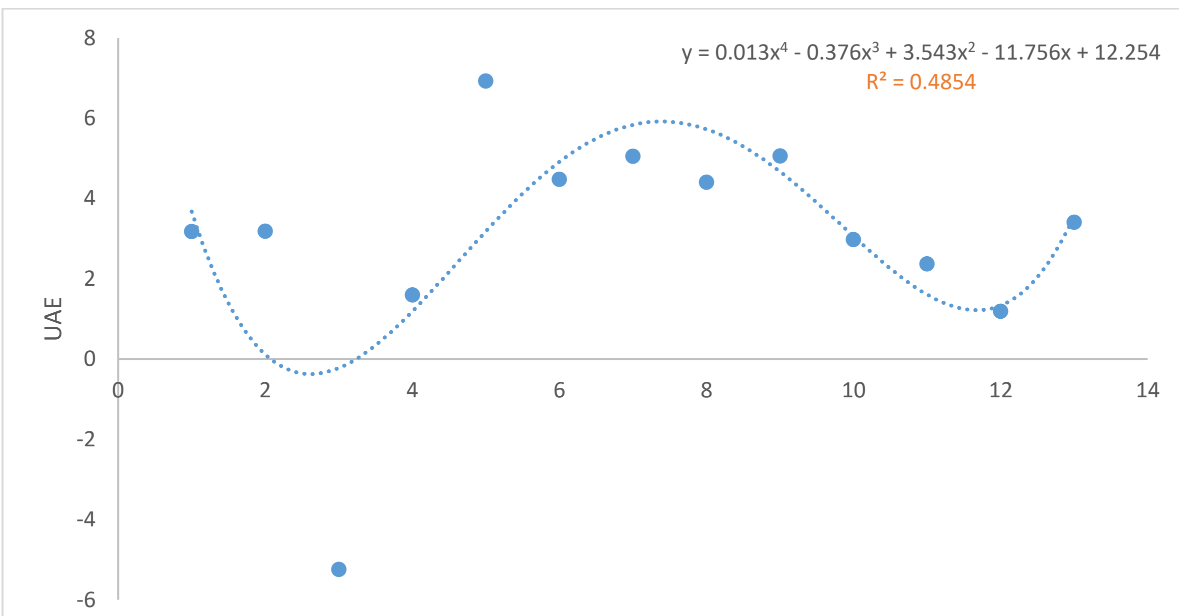


Figure 8.5 *GDP of UAE for 2007-2019*

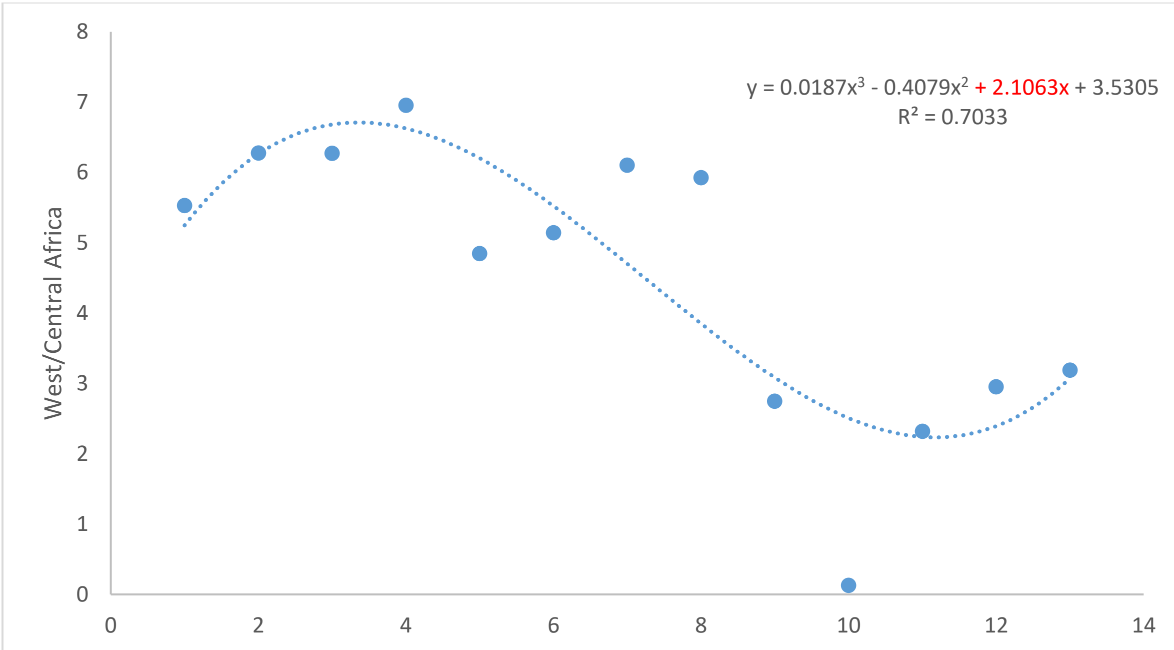


Figure 8.6 *GDP of West/Central-Africa for 2007-2019*

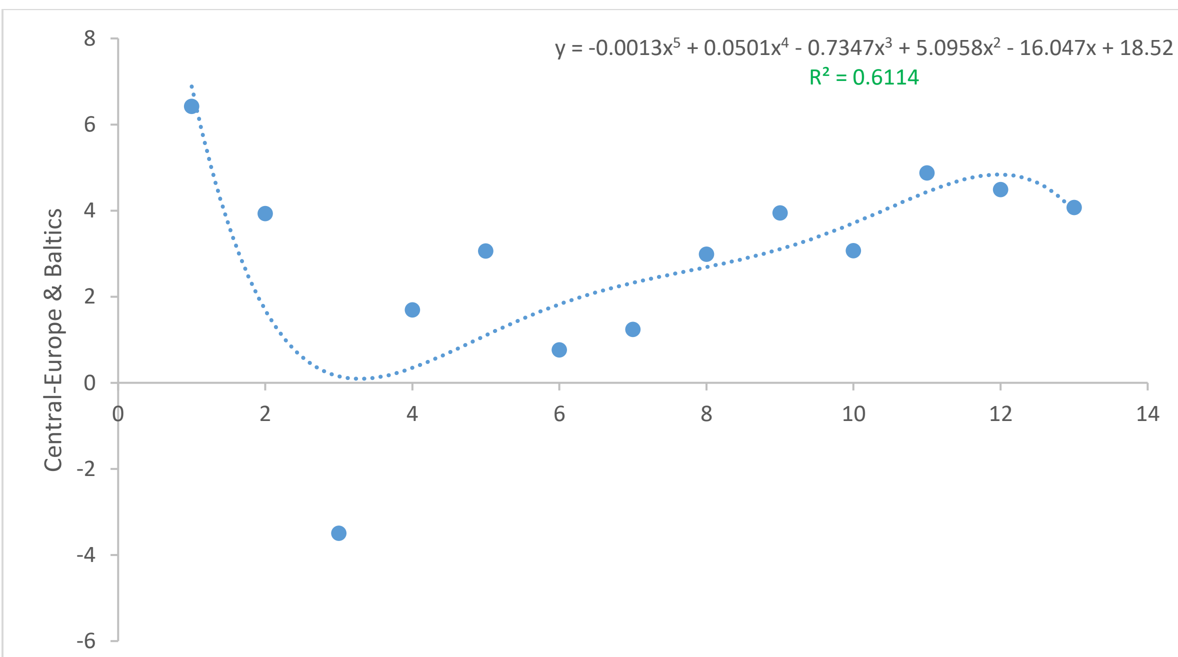


Figure 8.7 *GDP of Central-Europe & Baltics for 2007-2019*

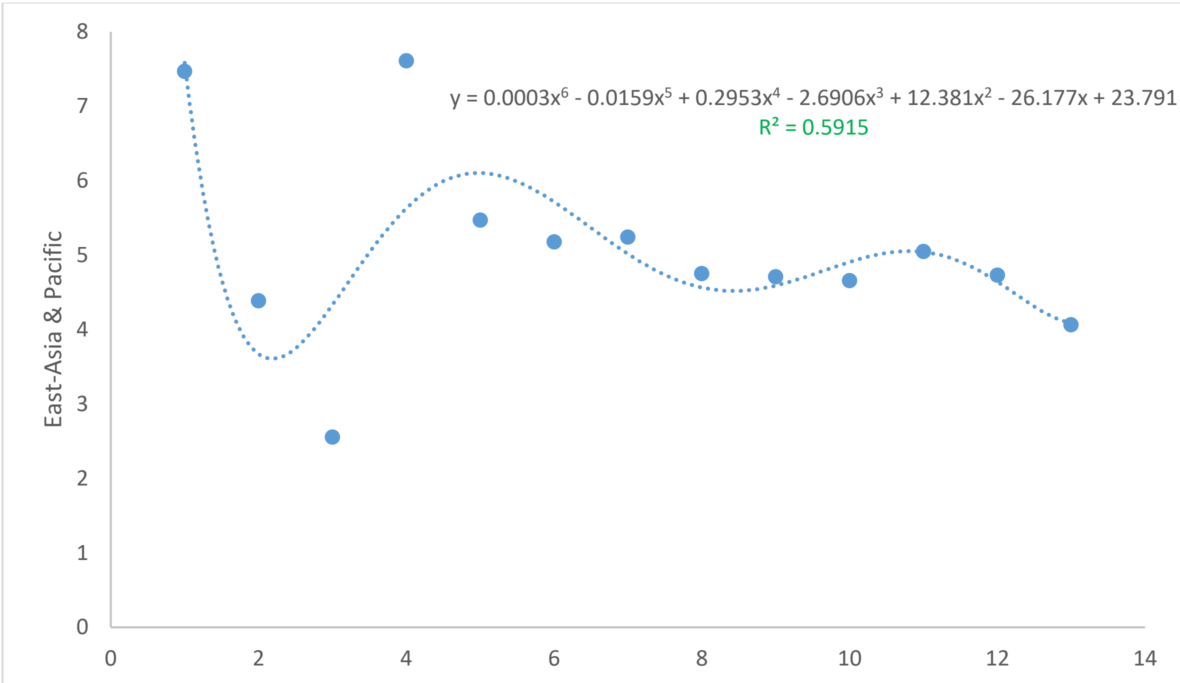


Figure 8.8 *GDP of East-Asia & Pacific for 2007-2019*

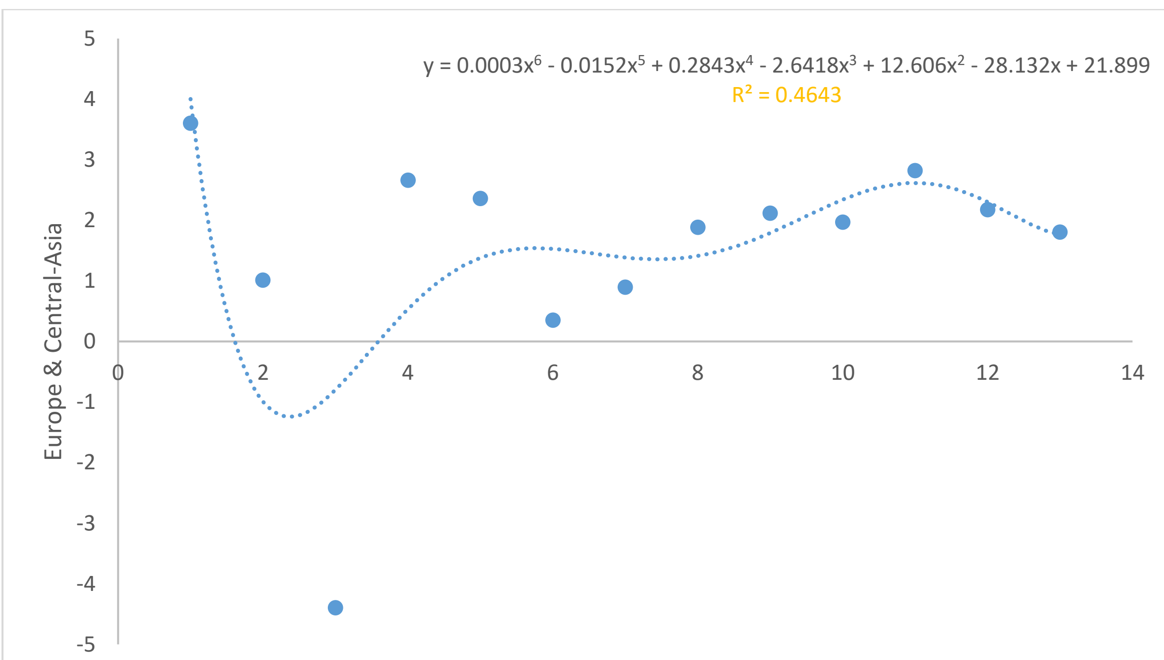


Figure 8.9 *GDP of Europe & Central-Asia for 2007-2019*

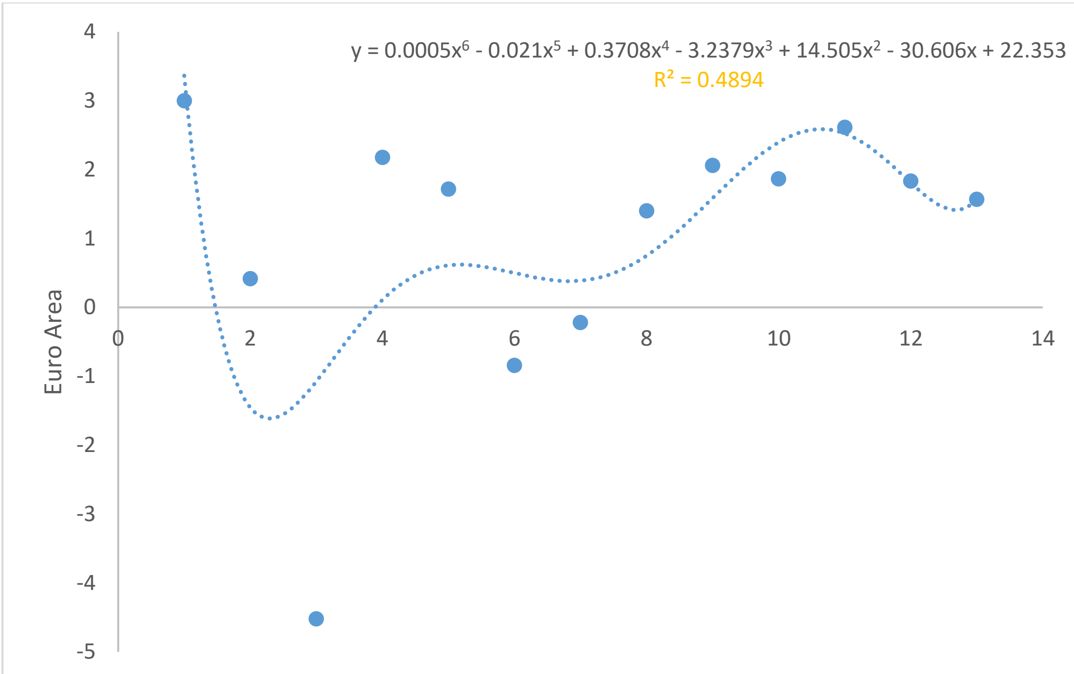


Figure 8.10 *GDP of Euro Area for 2007-2019*

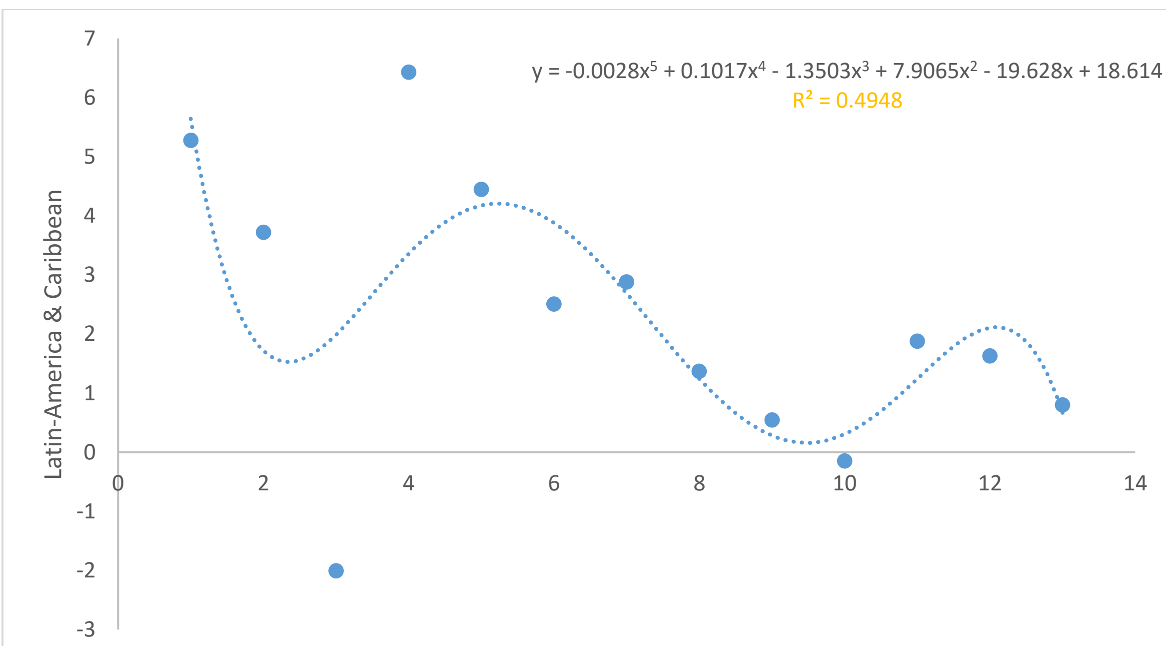


Figure 8.11 *GDP of Latin-America & Caribbean for 2007-2019*

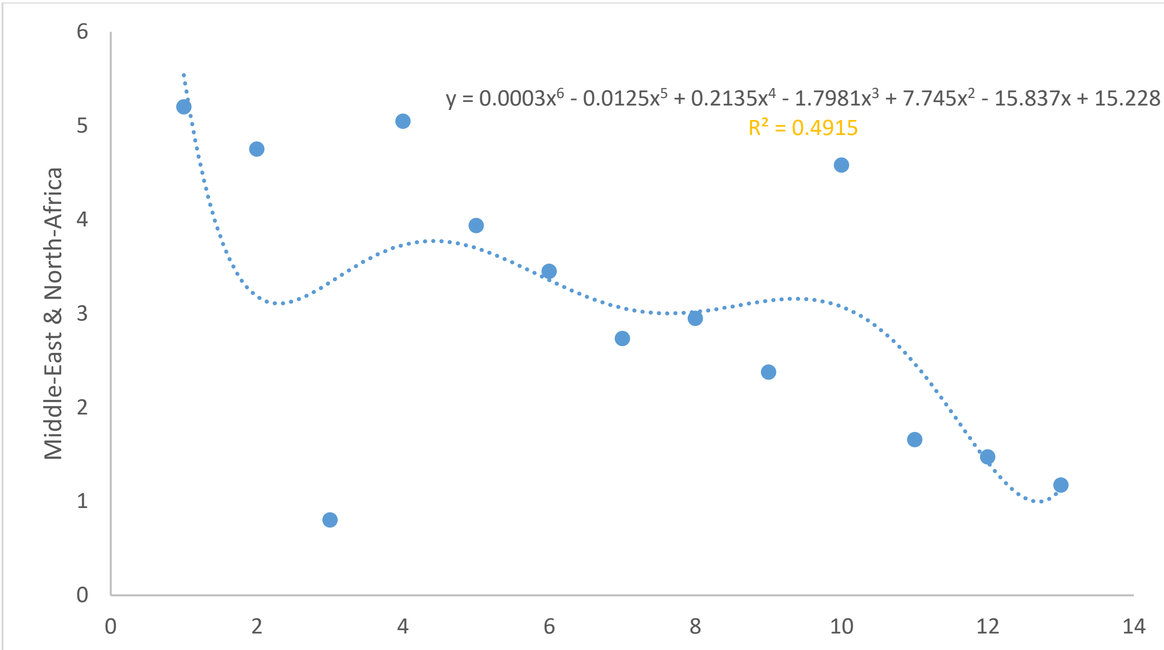


Figure 8.12 *GDP of Middle-East & North-Africa for 2007-2019*

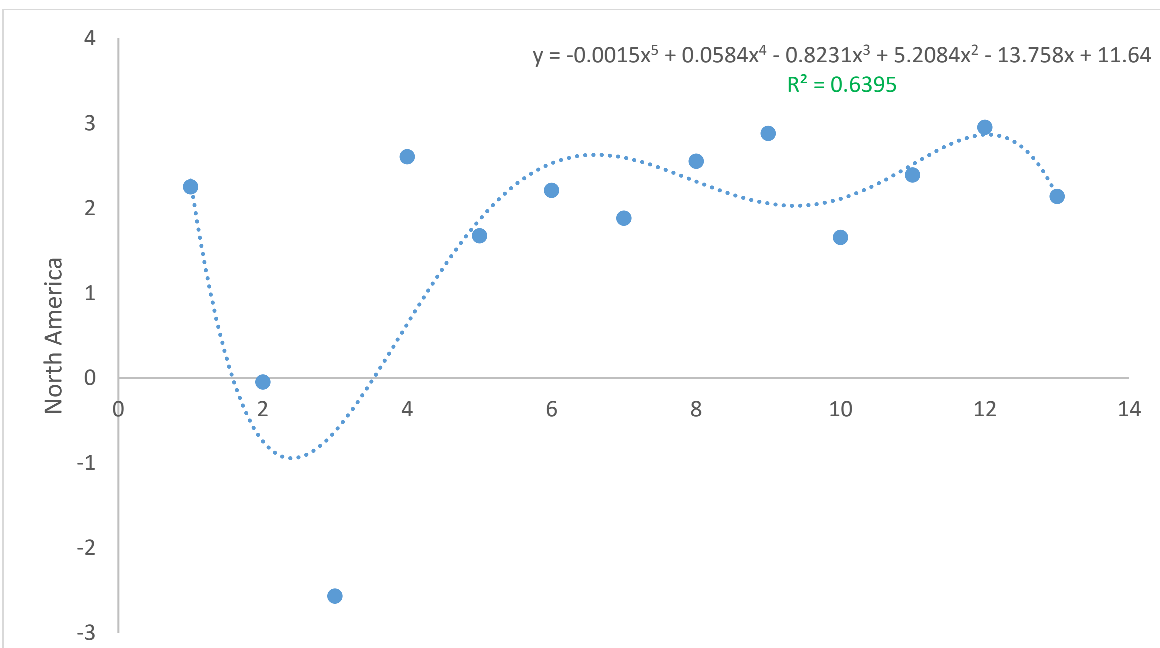


Figure 8.13 *GDP of North America for 2007-2019*

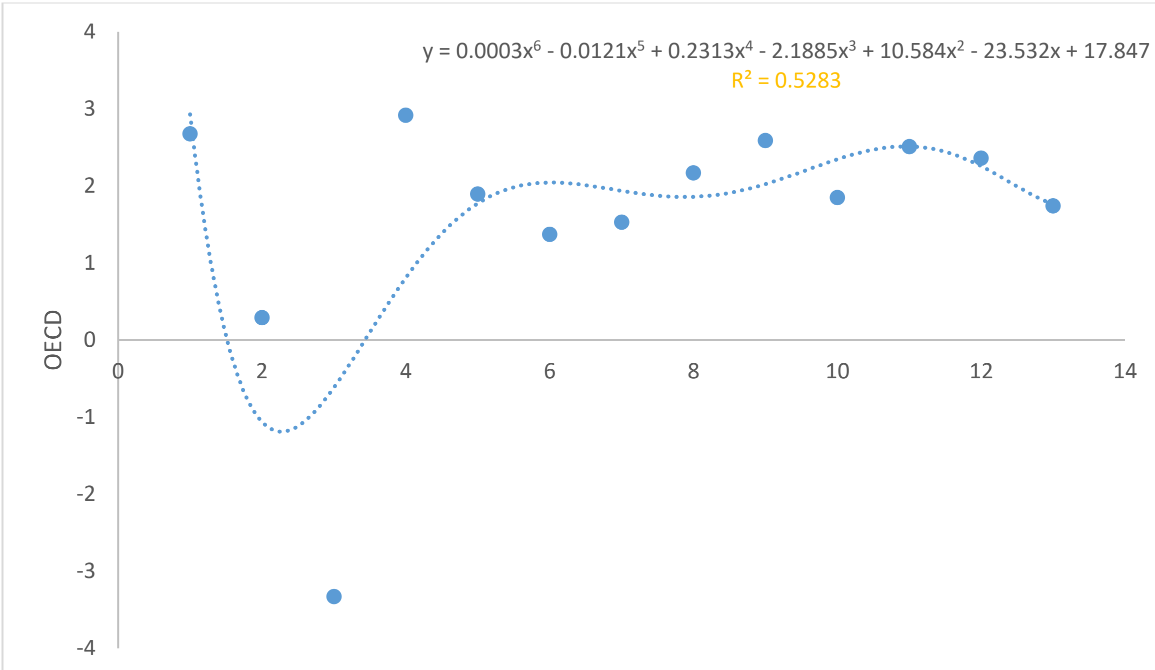


Figure 8.14 *GDP of OECD for 2007-2019*

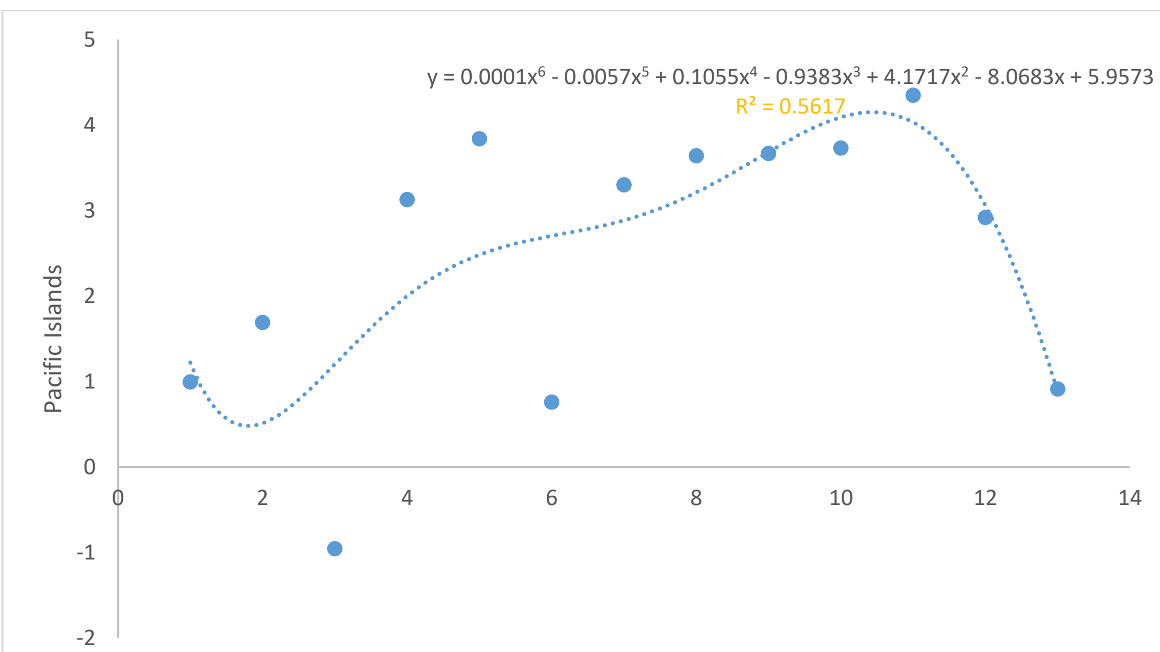


Figure 8.15 *GDP of Pacific Islands for 2007-2019*

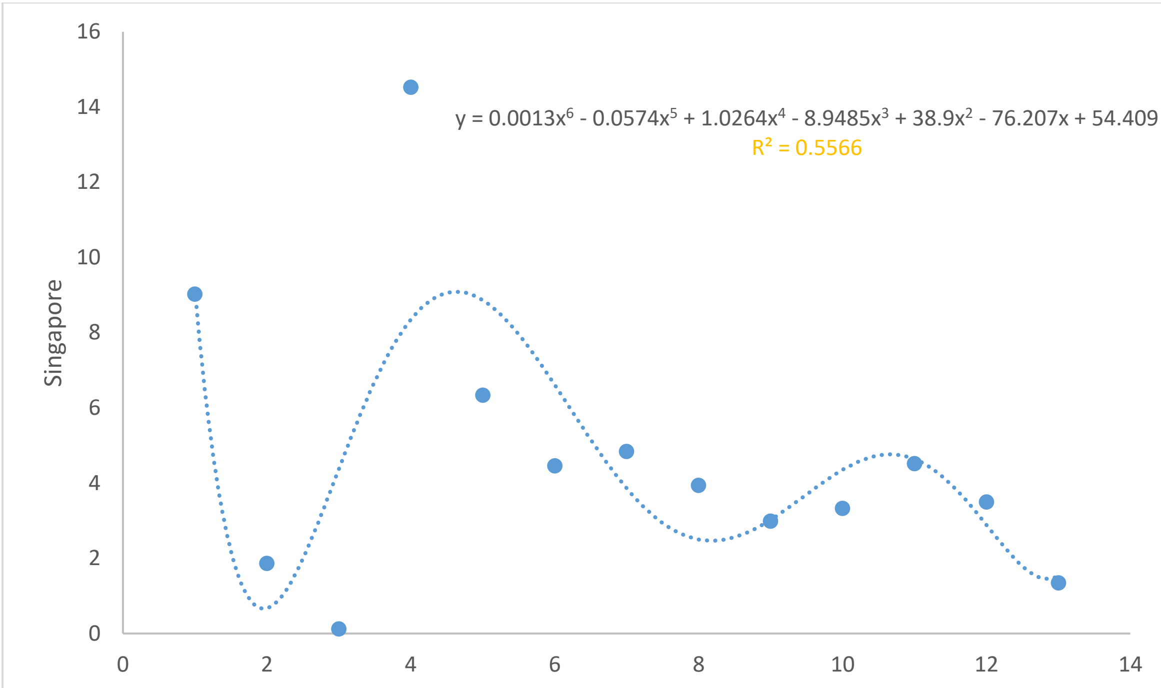


Figure 8.16 *GDP of Singapore for 2007-2019*

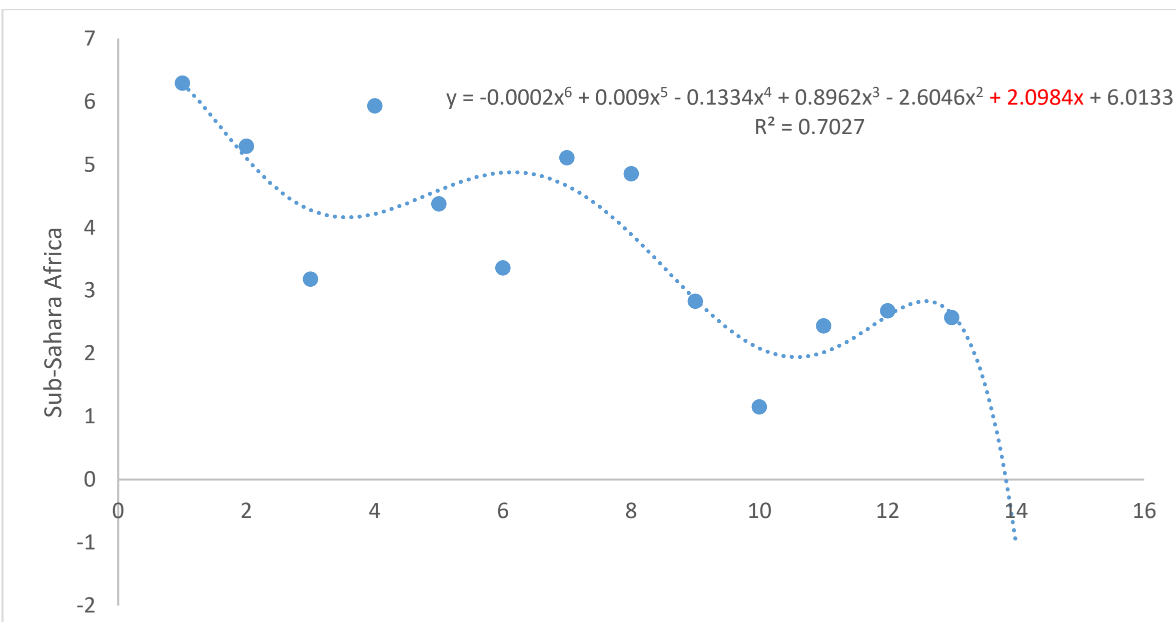


Figure 8.17 *GDP of Sub-Saharan Africa for 2007-2019*

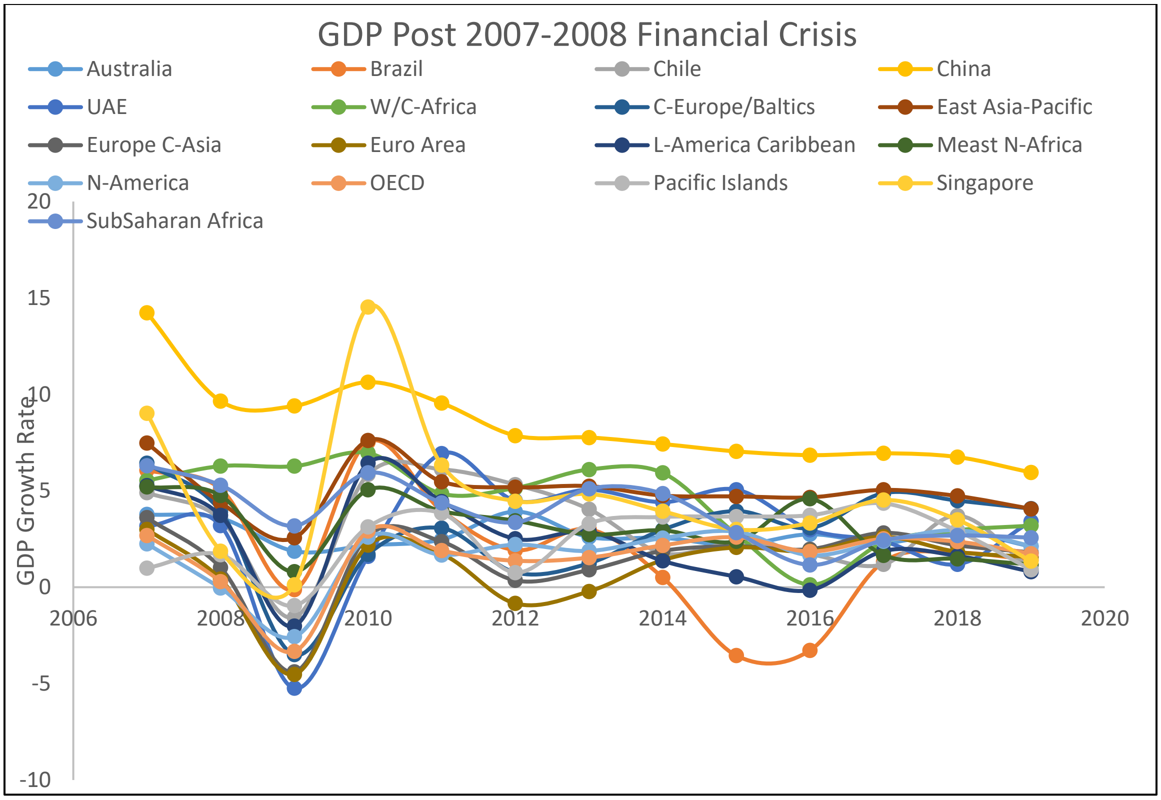


Figure 8.18 *Collective GDP Plots Across Countries/Regions for 2007-2019*

8.4 Analysis of Data

1. Satisfying both the mathematical criteria for J-Curve: Brazil, Chile, China, Central-Europe and Baltics, East Asia and Pacific, North America.

2. Likely displaying J-Curve behavior (R-squared less than the cot-off value of 0.6, between 0.48 and 0.6): UAE, Europe and Central-Asia, Euro-Asia, Latin America and Caribbean, Middle East and North Africa, OECD, Pacific Islands, Singapore.

3. Did not satisfy the Laguerre polynomial criteria: Australia, West/Central Africa, and Sub-Saharan Africa.

Country	Polynomial Criteria	R-squared Criteria
Australia	No	NA
Brazil	Yes	Yes
Chile	Yes	Yes
China	Yes	Yes
UAE	Yes	Maybe
West/Central Africa	No	NA
Central-Europe and Baltics	Yes	Yes
East-Asia Pacific	Yes	Yes

Country	Polynomial Criteria	R-squared Criteria
OECD	Yes	Maybe
Singapore	Yes	Maybe
Euro Area	Yes	Maybe
North America	Yes	Yes
Sub-Saharan Africa	No	NA
Europe Central-Asia	Yes	Maybe
Latin-America and Caribbean	Yes	Maybe
Middle-East and North-Africa	Yes	Maybe
Pacific Islands	Yes	Maybe

Table 8.2 *J-Curve Behavior of Countries/Regions post 2007-08 Financial Crisis.*

8.5 Key Results

a) Most countries/regions showed J-Curve type behavior, as expected.

b) Australia, Western/Central-Africa and Sub-Saharan Africa showed no J-Curve behavior. Why?

c) China showed minimal J-Curve effect.

d) Singapore had steepest recovery, maybe since it is a small country.

e) Brazil showed a second slump around 2015-16. Why?

CHAPTER IX:

CASE STUDY 5: J-CURVE AND GDP OF CROATIA

9.1 Economy of Croatia

We studied the J-Curve relation with four important economic parameters - growth in GDP, GNI, Trade Balance, and Manufacturing for over 60 years – of a large country like India with 1.38 billion population in Chapter 5. J-Curve behaviour was observed in the period post the 1991 economic liberalization (as expected).

It will be interesting to see if J-Curve phenomenon exists in the economic evolution of a much smaller country, which has recorded its GDP data over the past 25 years. As an illustrative example, we choose the country of Croatia, with a population of 4.05 million.

The economic reforms in Croatia were set in the mid-1990s. Since it would take some time for the effect to be seen – a typical J-Curve in the GDP can be expected over the 10+ years in the period 1996-2007.

The global financial crisis of 2007-2008 affected the economy of many countries. Since various remedial steps put in place take effect after a certain time lag, a J-Curve behaviour is again expected in the GDP over the period 2008-2019.

(“Croatia - Economy | Britannica”, n.d.) and (“Croatia”, n.d.) give some good background about Croatia’s economy.

9.2 GDP Growth Data of Croatia 1996-2019

(“GDP growth (annual %) - Croatia | Data”, n.d.)

Year	GDP	Year	GDP
1996	5.975681	2008	1.895444
1997	6.132025	2009	-7.2817
1998	2.205833	2010	-1.2507
1999	-0.87645	2011	-0.08524
2000	2.895501	2012	-2.27571
2001	3.032498	2013	-0.36312
2002	5.722074	2014	-0.34595
2003	5.525947	2015	2.525688
2004	4.149337	2016	3.532432
2005	4.312401	2017	3.414092
2006	4.938357	2018	2.90001
2007	4.912811	2019	3.481946

Table 9.1 *Croatia GDP 1996-2019*

9.3 Analysis of GDP Growth Data of Croatia 1996-2019

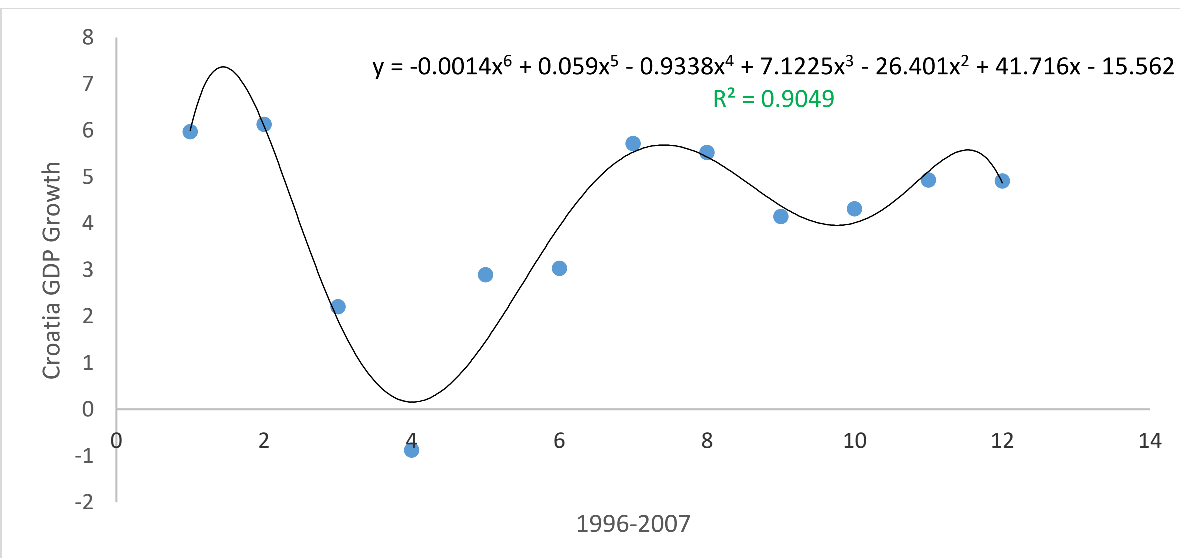


Figure 9.1 *J-Curve in Croatia GDP 1996-2007*

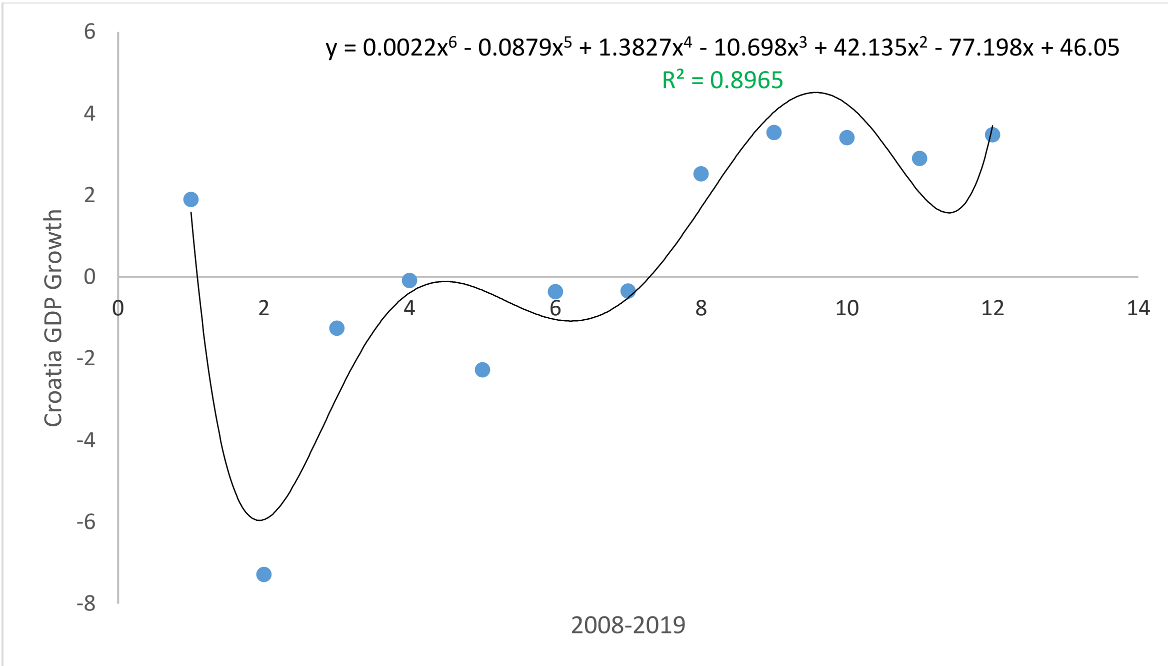


Figure 9.2 *J-Curve in Croatia GDP 2008-2019*

9.4 Key Results

As expected, mathematically validated J-Curve is observed in the GDP growth for both the periods studied:

1. 1996-2007: post-independence.
2. 2008-2019: post global financial crisis.

CHAPTER X:

SUMMARY, IMPLICATIONS AND RECOMMENDATIONS

10.1 Validating the Various Hypothesis

We outlined the various Hypotheses to be studied and validated in Chapter 3. The key results obtained in this research are validated vis-à-vis these Hypotheses.

H1: The differential equation for the S-Curve can be recast as a Riccati equation - *this was achieved.*

H2: The J-Curve can be described by a differential equation, which is also in the form of the Riccati equation – *the explicit form of the Riccati equation was derived*.

H3: A common mathematical formalism to the S-Curve, as well as the J-Curve, can be given in the framework of a general nonlinear 1st order Riccati equation – *the general form of the Riccati equation was given, and the equations for the S-Curve and J-Curve are special cases*.

H4: A suitable linear 2nd order differential can be found to describe the J-Curve (Appendix B) – in terms of the Laguerre polynomials. This gives the functional form for the J-Curve – *the 2nd order linear equation associated with the J-Curve Riccati equation is shown to be the Laguerre equation, and the functional form of the Laguerre polynomial is proposed as the equation to describe the J-Curve.*

H5: A quantitative process can be set up to mathematically verify any curve that will be classified as a J-curve - *two criteria to mathematically validate a J-Curve are given in terms of the a) form of the polynomial representation and b) statistical criteria in terms of a threshold for R-squared value*.

H6: Cases in which the J-curve is expected / observed in practice can be mathematically proven to show the behavior of the J-curve - *all 5 case studies* *were carried out and the phenomena of the J-Curve in all the expected scenarios were mathematically proven*.

H7: Functional form of medicine absorption in the body will be presented – *the equation for absorption of medicine is presented*.

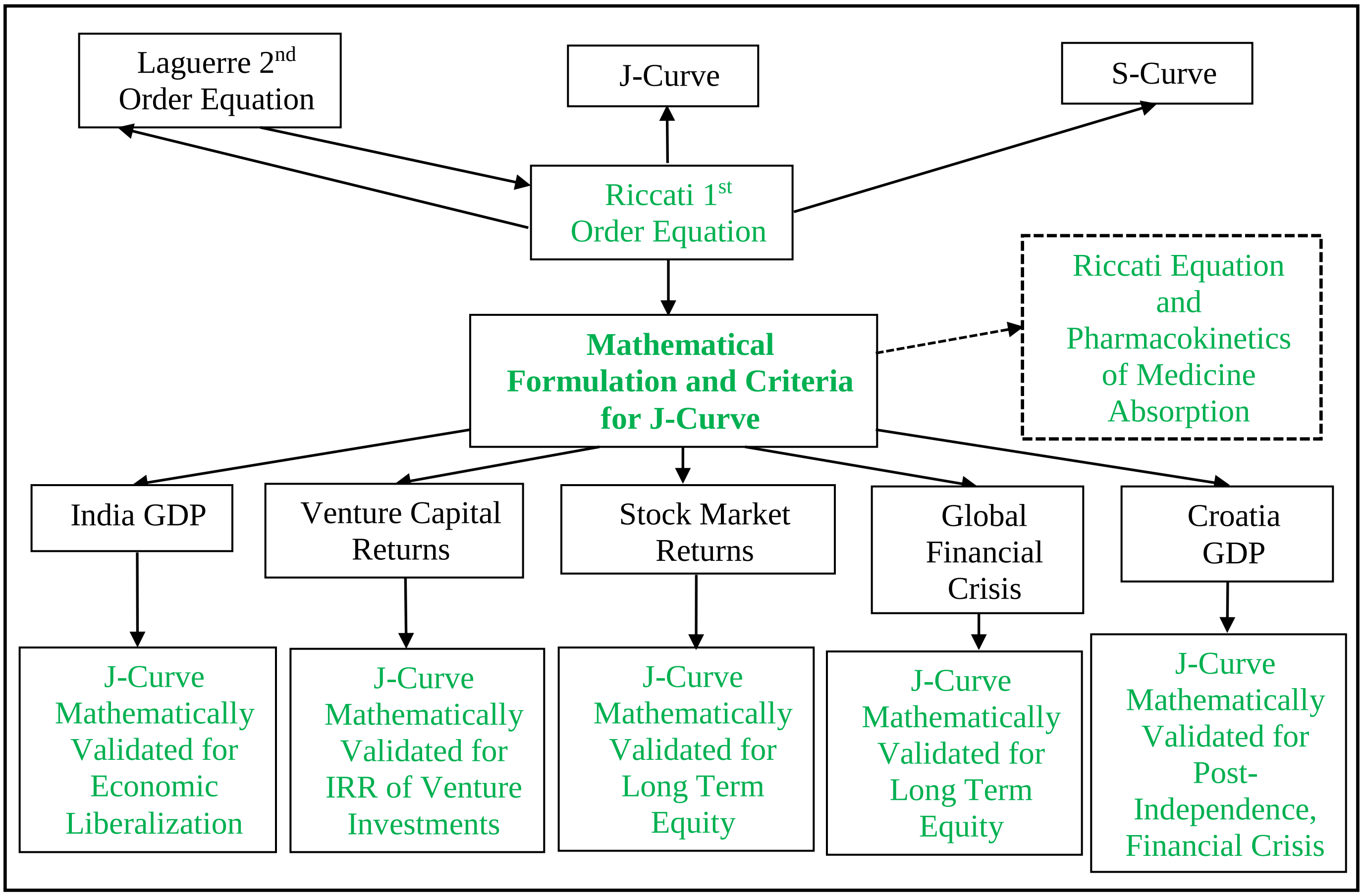


Figure 10.1 *Summary of Thesis Results*

10.2 Implications

a) While the S-Curve has a differential equation description, similar such description has not existed for the J-Curve. This has been resolved.

b) A differential formalism for the J-Curve has been presented for the first time. This gives scope for further formal studies of the J-Curve.

c) Solutions describing the J-Curve are given as (Laguerre) polynomials with coefficients having alternate signs. These criteria are used to mathematically validate if a system displaying J-Curve. This gives a formal mathematical framework for any curve to be classified as a J-Curve.

d) This formal analysis and approach paves the way for more/other mathematical techniques to be used to model and study important concepts in business, economics, finance, healthcare, etc.

10.3 Recommendations for Future Research

a) Various other systems in economics, politics, finance, healthcare, entrepreneurship, etc. need to be studied and mathematically validated for J-Curve behavior, as per the framework set up in this thesis.

b) Are nonlinear 1st order Riccati differential equations the only mathematical framework to describe the S-Curve and the J-Curves? Will higher order Riccati differential equations play a role in the description of the J-Curve?

c) Are polynomials with coefficients having alternating signs the only way to describe the J-Curve? Are there other equations that do so?

d) Are differential equations the only way to describe J-Curves? Are there other mathematical techniques to model the J-Curve?

e) Can further studies be made to validate that the pharmacokinetics of absorption of medicines in the body do indeed follow the equation given in Appendix E?

APPENDIX A

MATHEMATICAL PROPERTIES OF RICCATI EQUATION

NOTE: All the results mentioned in this section are well known results and taken from literature (Davis, 1975).

A general nonlinear 1st order Riccati differential equation is of the form:

$$\frac{dy(t)}{dt} + Q(t) * y(t) + R(t) * y(t)^2 = P(t) \qquad \text{A.1}$$

where y(t) is the dependent variable and t (time in our context) is the independent variable.

This **nonlinear 1st order** Riccati equation can always be converted into a corresponding **linear 2nd** order differential equation of the form:

$$R(t) * \frac{d^2u(t)}{dt^2} - \left(\frac{dR(t)}{dt} - Q(t) * R(t)\right) * \frac{du(t)}{dt} - P(t) * R(t)^2 * u(t) = 0 \qquad \text{A.2}$$

where

$$y(t) = \frac{\frac{du(t)}{dt}}{R(t) * u(t)} \qquad \text{A.3}$$

Similarly, any linear 2nd order linear differential equation

$$A(t) * \frac{d^2u(t)}{dt^2} + B(t) * \frac{du(t)}{dt} + C(t) * u(t) = 0 \qquad \text{A.4}$$

can be converted into a corresponding nonlinear 1st order Riccati equation

$$\frac{dy(t)}{dt} + \left(\frac{\frac{dR(t)}{dt}}{R(t)} + \frac{B(t)}{A(t)} \right) * y(t) + R(t) * y(t)^2 = -\frac{C(t)}{\big(A(t) * R(t)\big)} \qquad \text{A.5}$$

where

$$\frac{du(t)}{dt} = R(t) * y(t) * u(t) \qquad \text{A.6}$$

It is possible to choose R(t) such that

$$\frac{\frac{dR(t)}{dt}}{R(t)} + \frac{B(t)}{A(t)} = 0 \qquad \text{A.7}$$

Hence, for every linear 2nd order differential equation (A.4), we can get the corresponding nonlinear 1st order Riccati equation

$$\frac{dy(t)}{dt} + R(t) * y(t)^2 = P(t) \qquad \text{A.8}$$

where

$$R(t) = \exp(-\int \frac{B(t)}{A(t)} dt \qquad \text{A.9}$$

The reason for setting up this important equivalence between eqn. (A.1) and eqn. (A.2) and, vice versa, between eqn. (A.4) and eqn. (A.8) is that nonlinear equations are usually difficult to solve, whereas very many formal/analytical methods exist to solve linear 2nd order differential equations. Thus if a system is described by a nonlinear Riccati equation, one can solve the equivalent linear 2nd order equation whose solution will also be the solution of the original Riccati equation.

Conclusion: For every nonlinear 1st order Riccati differential equation, there is a corresponding linear 2nd order differential equation, and vice versa. This very important relationship between (A.4) and the corresponding (A.8) will be used to find a suitable mathematical model for the J-Curve via a particular type of 1st Riccati equation, as well as the analytical form of the solution which will describe the J-Curve.

APPENDIX B

MATHEMATICAL PROPERTIES OF ORTHOGONAL POLYNOMIALS

NOTE: The results mentioned in this section are well known and taken from literature (Balakrishnan, 2020).

2nd order linear ordinary differential equations play a very important many areas of engineering (e.g. fluid mechanics, solid mechanics, telecommunications, heat transfer, etc.) and sciences (e.g. quantum mechanics, electromagnetic theory, classical mechanics, etc.).

Many of these equations belong to a special category whose solutions are what are known as **special functions** and **orthogonal polynomials** (which are a part of the "special functions"), all of which have been extensively studied (Andrews *et al.*, 1999; Lebedev and Silverman, 1972; Vilenkin and Klimyk, 2013).

The orthogonal polynomials are related to a very important concept of **hypergeometric differential equation**, which has the form:

$$A(t) * \frac{d^2u(t)}{dt^2} + B(t) * \frac{du(t)}{dt} + C * u(t) = 0 \qquad \text{B.1}$$

where

$$A(t) = a * t^2 + b * t + c, and\ B(t) = d * t + e \qquad \text{B.2}$$

and a, b, c, d, and e are constants.

For particular values of constant C in (B.1) given by

$$Cn = -(n(n-1) * a + n * d) for\ n = 0,1,2 \qquad \text{B.3}$$

The solutions of (B.1) are orthogonal **polynomials of the hypergeometric type**. For the classical orthogonal polynomials, there are 3 possible cases for the range of the independent variable 't':

i) $-\infty\ to + \infty$, corresponding to **Hermite** polynomials

ii) $0\ to\ \infty$, corresponding to **Laguerre** polynomials

iii) $-1\ to + 1$, corresponding to **Jacobi** polynomials, which include 3 special polynomials named **Gegenbauer**, **Chebyshev** (of Types I and II), Legendre (of Types I and II), and **Associated Legendre**.

This can be summarized in the below Figure B.1 below.

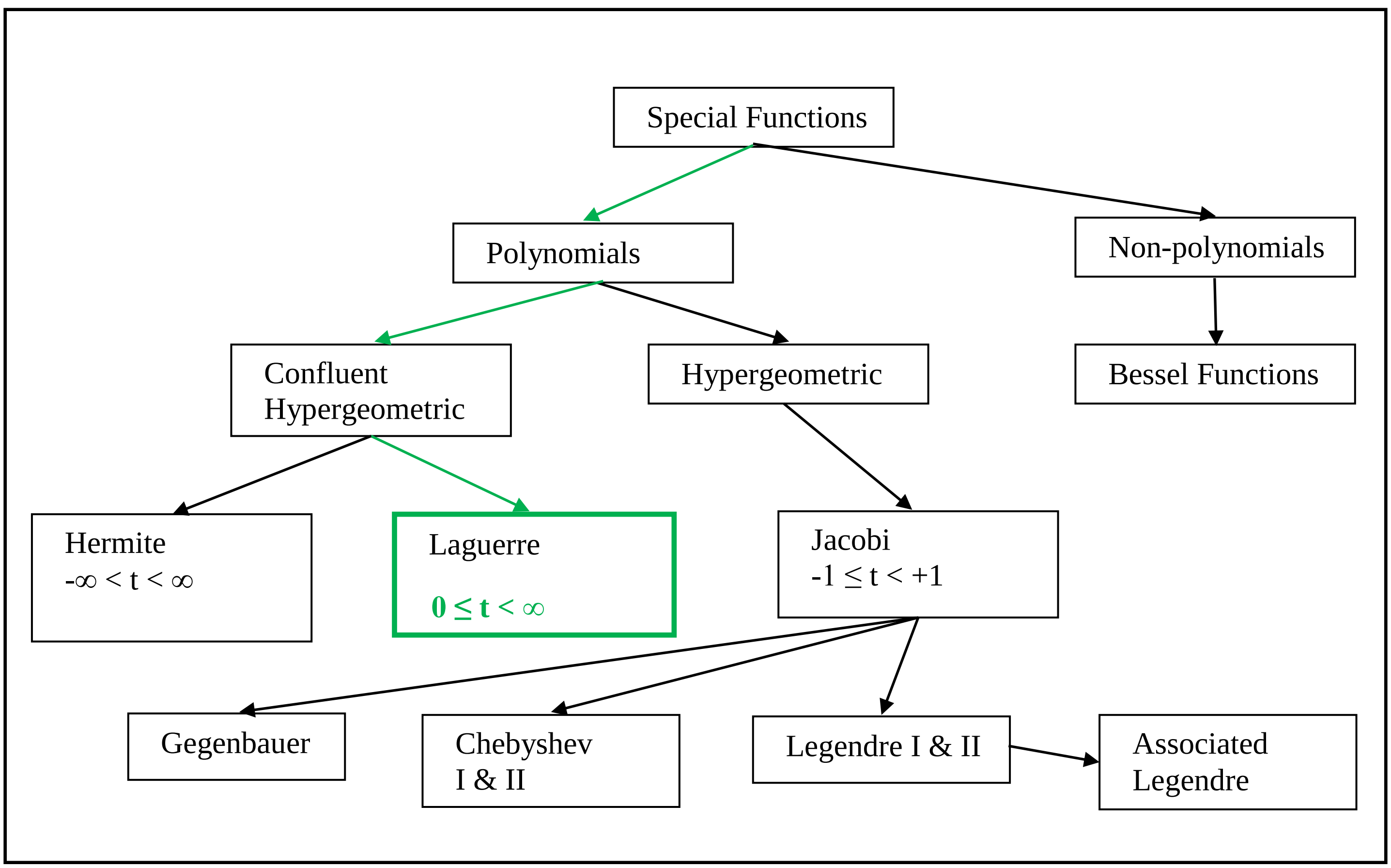


Figure B.1 *Summary of Special Functions*

Since we are interested in studying J-Curve phenomenon in various systems with a “Cause-and-Effect” characteristic where a certain cause at t=0 causes an effect for future times – we shall be interested in the independent variable ‘t’ taking the range 0 to ∞. Thus, due to physical considerations, we shall be exploring only the **Laguerre** polynomials only (amongst the entire family of polynomials), and propose that they can give a suitable framework to mathematically describe the J-Curve.

Some specific and interesting properties of the Laguerre polynomials will be studied in Appendix C.

Conclusion: In the search for a suitable linear 2nd order differential equation, and its solution, to describe the J-Curve, the potential candidate has been identified as the differential equation for the Laguerre polynomial. It will be presented in detail in Appendix C.

APPENDIX C

MATHEMATICAL PROPERTIES OF LAGUERRE POLYNOMIALS

NOTE: The results mentioned in this section are well known and taken from literature (Arfken and Weber, 1999).

The differential equation for the Laguerre polynomial is obtained from the hypergeometric differential equation (B.1) by putting A(t) = t, B(t) = (1-t), and C = n (where n is an integer) to get

$$t * \frac{d^2u(t)}{dt^2} + (1 - t) * \frac{du(t)}{dt} + n * u(t) = 0 \qquad \text{C.1}$$

The Laguerre polynomials are denoted by L_n (the n^{th} order Laguerre polynomial). The first few polynomials are given by

$L_0 (t) = 1$
$L_1 (t) = -t + 1$
$L_2 (t) = t^2 - 4t + 2$
$L_3 (t) = -t^3 + 9t^2 - 18t + 6$
$L_4 (t) = t^4 - 16t^3 + 72t^2 - 96^t + 24$
$L_5 (t) = -t^5 + 25t^4 - 200t^3 + 600t^2 - 600t + 120$
$L_6 (t) = t^6 - 36t^5 + 450t^4 - 2400t^3 + 5400t^2 - 4320t + 720$

The following recursive relation obtains higher order polynomials

$$L_{n+1} (t) = [(2n + 1 - t) * L (t) - n * L_{n-1} (t)] / (n + 1) \qquad \text{C.2}$$

The plots of the initial few Laguerre polynomials are given below (Weisstein, n.d.) in Figure C.1 below:

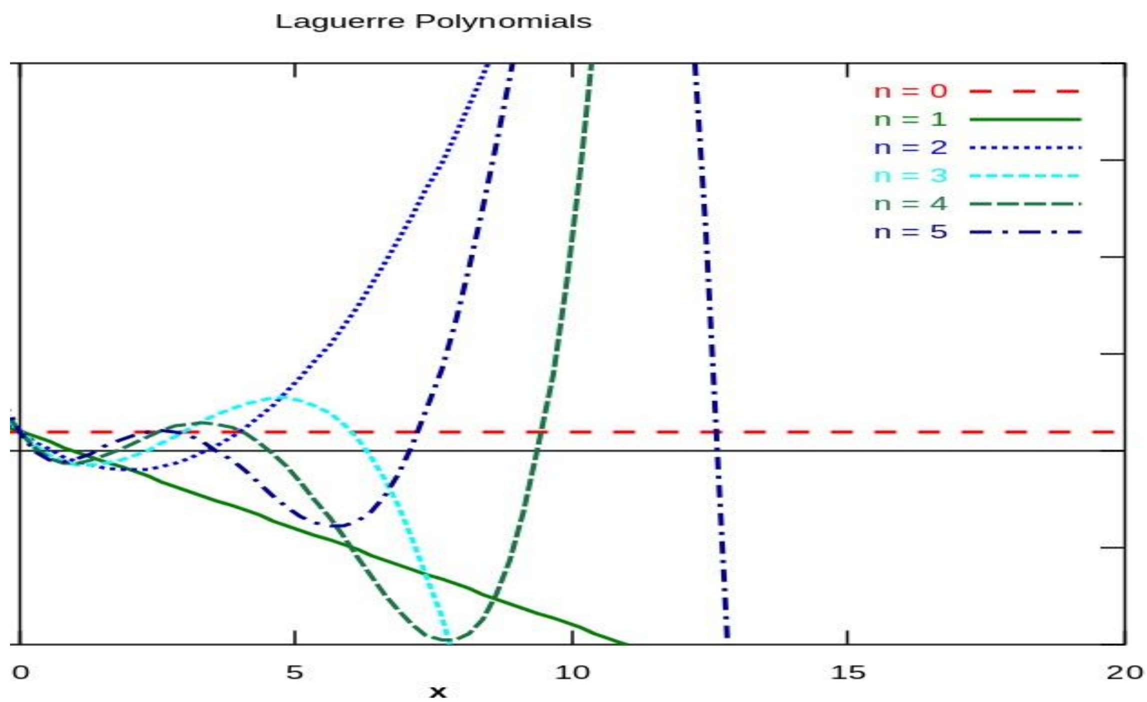


Figure C.1 *Laguerre Polynomials*

It can be seen that L_3 (t), and higher order polynomials, all satisfy the typical characteristic of the J-Curve. One of the most important characteristic/property of the Laguerre polynomials is that they are described by polynomials with coefficients alternating in signs.

Usually, the number of inflexions in the curve dictate the order of the polynomial – e.g. if the J-Curve has 2 inflexions, it is natural to choose 3^{rd} order Laguerre polynomial L_3.

Conclusion: Amongst all the hypergeometric differential equations and their solutions in terms of orthogonal polynomials, we identified the Laguerre polynomial as the most promising candidate for describing the J-Curve. Further aspects and details are described in Chapter 4.

APPENDIX D

PHYSICAL PROPERTIES OF RICCATI DIFFERENTIAL EQUATION

We proposed the differential equation for the J-Curve as

$$t * \frac{dy(t)}{dt} + \mathrm{e}^{\mathrm{t}} * y(t)^2 = -t * e^{-t} \qquad \text{D.1}$$

It is important to study what the various terms in this equation represent.

Towards this, a similar form of the equation which has been one of the most studied equation in physics and engineering is given below - that of a particle falling under gravity in a viscous damping medium:

$$m * \frac{dy(t)}{dt} + D * y(t)^2 = F(t) = m * g \qquad \text{D.2}$$

Here y(t) is the velocity of the particle (to be solved), m (constant) is the mass of the particle, g (constant) is the acceleration due to gravity and D (constant) is the damping constant due to the viscous medium in which the particle is falling.

In a more general analogy, the above equation can be generalized to

$$I(t) * \frac{dy(t)}{dt} + D(t) * y(t)^2 = F(t) \qquad \text{D.3}$$

This equation describes a system of time-dependent and varying inertia I(t) (which can be interpreted as the mass for the example of the particle) evolving in a damping environment characterized by the damping parameter D(t), under the influence of environmental/ecosystem forces F(t).

Conclusion: Eqn. (D.3) will be the form used to set up the differential equation for describing the J-Curve. The associated I(t), D(t) and F(t) (as seen in eqn. (D.1)) will give a physical interpretation to dynamics of any system which manifests the J-Curve behavior. The solutions describing the J-Curve will be in the form of Laguerre polynomials described in Appendix C.

APPENDIX E:

RICCATI EQUATION AND HEALTHCARE

a) Interpretation of Riccati Equation to Healthcare

The Riccati differential equation (eqn. (4.5))

$$t * \frac{dy(t)}{dt} + \mathrm{e}^{\mathrm{t}} * y(t)^2 = -n * t * e^{-t} \qquad 4.5$$

can be given a different perspective in the case of healthcare:

i) The dependent variable y(t) is taken to be the health of the patient,

ii) I(t) = t is the inertia (mass) of the patient,

iii) D(t) = e^{t} is the growth of illness which is affecting the health y(t) of the patient,

iv) F(t) = $t * e^{-t}$ is the effect of the external medication on the patient's health y(t).

It is important to see if the interpretation of the effect of medication in the human body given by F(t) = $t * e^{-t}$ does indeed agree with the what has been observed clinically.

Towards that, two important studies are analyzed here: a) the effect of statins in the treatment of high cholesterol (Neuvonen *et al.*, 2008), b) the effect of medication in the treatment of hypertension (Kiriyama *et al.*, 2016)

b) Interpretation of Drug Absorption Experiments

The absorption of statins (for cholesterol) (Neuvonen *et al.*, 2008, p. 466) and antihypertensive medication (for hypertension) (Kiriyama *et al.*, 2016, p. 25) are shown below.

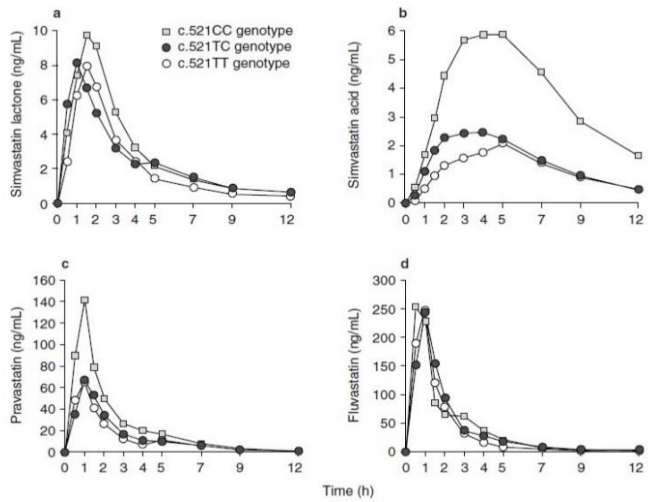


Fig. 1. Effect of the solute carrier organic anion transporter family member 1B1 *(SLCO1B1)* c.521 genotype on the disposition of HMG-CoA reductase inhibitors (statins). Mean plasma concentrations of (**a**) simvastatin lactone and (**b**) active simvastatin acid after a 40-mg dose of simvastatin, (**c**) pravastatin after a 40-mg dose of pravastatin, and (**d**) fluvastatin after a 40-mg dose of fluvastatin. The statins were given on separate occasions to the same group of 32 healthy volunteers with different *SLCO1B1* c.521 genotypes.[24,25]

Figure E.1 *Absorption of Statins (Neuvonen et al., 2008, p. 466)*

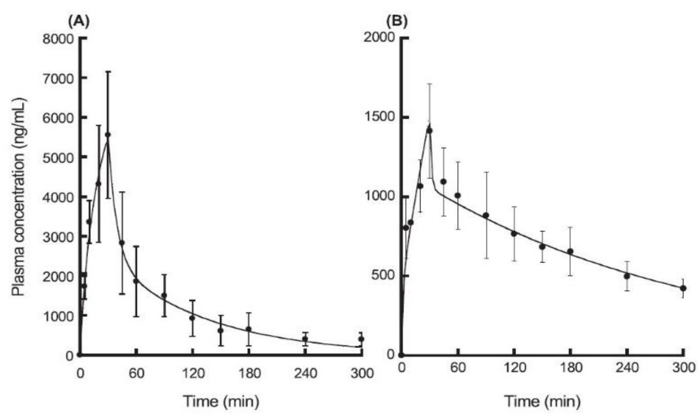


Fig. 4. Fitted and observed plasma concentrations of nifedipine (A) and propranolol (B) after iv infusion to SHRs. Drug doses of nifedipine and propranolol were 2.0 and 30 mg/kg, respectively. Fitted curves were both obtained according to the 2-compartment PK model. Each point represents the mean ± SD of the data obtained from 4 to 5 experiments.

Figure E.2 *Absorption of Antihypertensive Medication (Kiriyama et al., 2016, p. 25)*

It is interesting to observe that the absorption of drugs follow the curve given below:

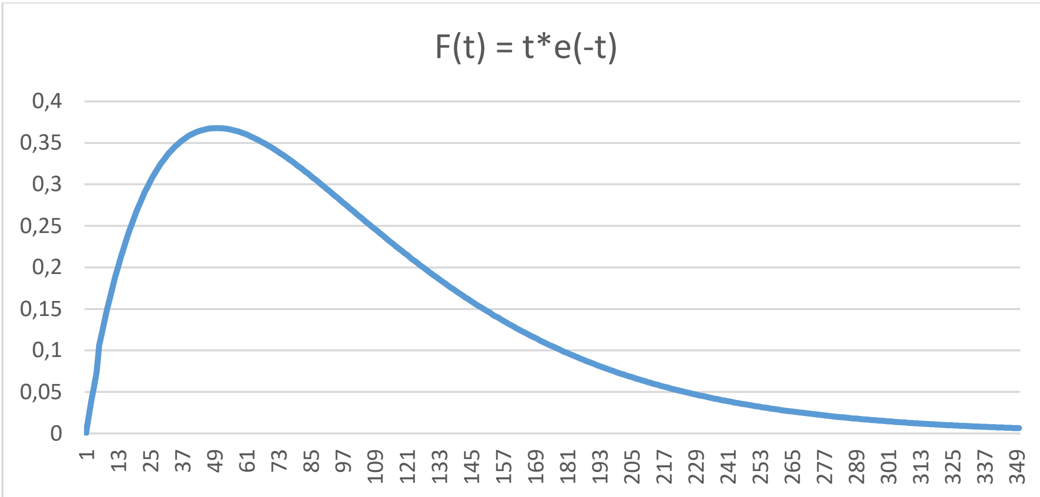


Figure 4.1 *F(t) = t*exp(-t)*

It was also observed in Chapter IV that the peak occurs at around 14% of the total time duration of the effect of the external force, and this seems to reasonably agree with the peak absorption time observed for both the anti-cholesterol and anti-hypertensive drugs.

Based on this observation, we postulate that the equation for the drug absorption A(t) is to be taken as

$$A(t) = t * e^{-t} \tag{4.5}$$

This is another important result arising out of describing via the Riccati differential equation.

c) J-Curve in Healthcare

Given the good relationship between the "external force" interpreted as the external medication and the absorption of the drug, it is natural to ask if the rest of the Riccati equation framework follows through – i.e. is the J-Curve observed in healthcare too.

There have been studies quoted about J-Curve and Hypertension (Banach and Aronow, 2012; Dudenbostel and Oparil, 2012; Williams, 2009). Similarly, J-Curve has also been observed with respect to cholesterol (Lange *et al.*, 1999; Matsuzaki *et al.*, 2002). However, since it has not been possible to access actual pharmacokinetic data for the levels of the drugs in the body as a function of time, it has not been possible to carry out curve fitting to see if the J-Curve observed is actually mathematically validated as outlined in Chapter IV.

Conclusions

The usefulness of modelling the J-Curve via the Riccati differential equation is clearly demonstrated in the context of healthcare.

The Force term in the Riccati equation F(t) = $t * e^{-t}$ clearly describes the time dependence of the absorption of medicines in the body. The peak of absorption and the asymptotic tapering off of the medicine observed in the case of hypertension and hypercholesterolemia agree with the predictions of the Force function F(t) = $t * e^{-t}$.

Since this Riccati equation is expected to result in J-Curve behavior, it is comforting to see that J-Curve behavior has indeed been observed in the study of medication for hypertension and cholesterol.

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